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Perturbative QED and QCD at Finite Temperatures and Densities

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The finite temperature and density QED and QCD are discussed from the perturbative viewpoint. A comparison between Abelian QED and non-Abelian QCD is made at every step. The calculation of the thermodynamic potential is performed up to $\alpha^2 \ln \alpha$, allowing the masses of the fermions to be arbitrary. The equation of state for QCD plasma is obtained and the phase transition to the hadronic phase is discussed.

1. INTRODUCTION

It is widely accepted that the known forces between elementary particles can be described by gauge theories. The strong interaction is described by a non-Abelian gauge theory, quantum chromodynamics (QCD) (for a general review, see Marciano and Pagels, 1978), while the electromagnetic interaction is governed by an Abelian theory, Quantum Electrodynamics (QED), which can be considered as the electromagnetic part of the low-energy limit of the Weinberg-Salam model (Weinberg, 1967; Salam, 1968) describing the unified electroweak forces. The Weinberg-Salam model is supposed to unify further with QCD to a grand unified theory (see, for example, Langacker 1981).

Finite temperatures and densities have provided an interesting framework for studying gauge theories. It has been shown (Collins and Perry, 1975), by using renormalization group arguments, that strong coupling approaches zero when the temperature or density increases. Thus, at high temperature or density QCD matter approaches a free gas of quarks and gluons, and hence perturbation theory becomes reasonable. On the other hand, the electromagnetic coupling increases with increasing energy.

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However, it is expected that grand unification takes place before the QED coupling becomes too large, preventing the reliable use of perturbation theory.

The electron gas in metals and the charged plasma in the ionosphere are examples where finite temperature and density QED can be applied, although its nonrelativistic limit, i.e., the quantum many-body theory (Fetter and Walecka, 1971), is sufficient in most contexts. The places where one might find a high enough density or temperature so that QCD plasma exists are more exotic. One is the cores of neutron stars, where the density is much greater than the nuclear density (Baym and Chin, 1976; Keistler and Kisslinger, 1976; Freedman and McLerran, 1978; Kisslinger and Morley, 1978, 1979). Another situation where such extreme conditions existed, was the early universe (Linde, 1979; Shyryak, 1980; Kalashnikov and Klimov, 1981; Suhonen, 1982; Toimela, 1983a; Cook and Mahanthappa, 1984; Schramm and Olive, 1983; Witten, 1984). When the universe was younger than about 10^{-5} sec, the temperature was comparable to nucleon rest energies, and quarks were liberated. Furthermore, such conditions can possibly be created in the laboratory by means of heavy ion collisions (for the present status see Jacob and Satz, 1982).

The history of the field theory at finite T and μ goes back to the fifties. The pioneering work was done by Matsubara (1955). The theory was further elaborated especially by Fradkin (1965). Gell-Mann and Brueckner (1957) studied the non-relativistic electron gas, while the relativistic case was discussed by Akhiezer and Pelerminsky (1960). The extension of the finite temperature formalism to general gauge theories was discussed by Bernard (1979), Dolan and Jackiw (1974), and Weinberg (1974).

The purpose of the present paper is twofold. Firstly, we shall present some new results corresponding to more accurate calculation of the thermodynamic potential of the $SU(N)$ gauge theory including massive fermions. Secondly, we want to give a review of the area of the $SU(N)$ theory at finite temperatures and densities, in order to clarify several points that, in the author's opinion, have not been expressed in the literature in the clearest way possible, and also in order to make the discussion in the paper more self-explanatory.

The rest of this paper contains the following: The notation used in the text is given at the end of this section. In Section 2 we discuss the general formalism of the field theory at finite T and μ. An expression for the partition function as a functional integral over the periodic (in the imaginary time) boson fields and antiperiodic fermion fields is derived. Furthermore the formal technique to work out the Fourier sums, arising from the periodicity of the space is discussed. In Section 3 we study the polarization tensor in both cases, $T \neq 0$ and $T = 0$. The interesting infrared limit is examined in more detail. In Section 4 the thermodynamic potential is calculated up to the order $o(g^4 \ln g)$, allowing the fermion masses to be arbitrary. The general formulas are then examined at various limits. In Section 5 we study the ground state of a fermion gas. In the arbitrary mass case the zero-temperature limit of the thermodynamic potential is evaluated up to $O(g^4 \ln g)$ and a nonrelativistic expansion for it is obtained. The equation of state is studied in Section 6. Moreover (in QCD), the phase transition to the hadronic phase is discussed. Lastly, in Section 7, we give a short summary and outlook.

1.1. Notations

The units are standard $\hbar = c = k_B = 1$.

The metric used is $g_{\mu\nu} = (1, -1, -1, -1)$, except in Sections 3.3 and 5, where the Euclidean metric $(1, 1, 1, 1)$ was found to be more convenient.

The Dirac γ matrices satisfy the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

2. FIELD THEORY AT FINITE TEMPERATURES AND DENSITIES

2.1. Partition Function

Our starting point for studying the thermal properties of any theory is the grand partition function

$$Z = \mathrm{Tr}\exp\left[-\beta\left(\hat{\mathscr{H}} - \sum_f \mu_f \hat{\mathscr{N}}_f\right)\right] \tag{1}$$

Here $\hat{\mathscr{H}}$ is the Hamiltonian operator, $\hat{\mathscr{N}}_f$ and μ_f are the number operator and the chemical potential, of particles of type f. β is the inverse temperature. From the thermodynamic potential, defined by

$$\Omega = -\frac{1}{\beta V}\ln Z \tag{2}$$

we get the thermodynamic quantities like the pressure p, the entropy density s, the average number density n_f and the energy density ε, by the equations

$$p = -\Omega \tag{3}$$

$$s = -\frac{\partial\Omega}{\partial T} \tag{4}$$

$$n_f = -\frac{\partial\Omega}{\partial\mu_f} \tag{5}$$

$$\varepsilon = \Omega + Ts + \sum_f \mu_f n_f \tag{6}$$

[Our definition for the thermodynamic potential Ω, equation (2), differs from the usual one by a factor of $1/V$.] The thermal expectation values of those physical observables that cannot be derived from the thermodynamic potential are still needed for the complete description of the theory. They are defined by

$$\langle\hat{\mathcal{O}}\rangle = \frac{1}{Z}\operatorname{Tr}\left[\hat{\mathcal{O}}\exp\left[-\beta\left(\hat{\mathcal{H}} - \sum_f \mu_f \hat{\mathcal{N}}_f\right)\right]\right] \tag{7}$$

In order to be more specific we shall restrict our discussion from now on to the $SU(N)$ gauge theory, including the appropriate fermion fields. However, all that we say in this section is applicable also to QED by just changing the $SU(N)$ gauge group to $U(1)$. The $SU(N)$ Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_f \bar{\psi}_f(i\not{D} - m_f)\psi_f \tag{8}$$

where the field tensor F and the covariant derivative D are defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{9}$$

$$D_\mu = \partial_\mu + igA_\mu \tag{10}$$

A problem arises, when we look at the gauge theories: the definition of the partition function, equation (1), depends on which gauge it is evaluated (Bernard, 1974). This problem is related to the appearance of nonphysical particles in some gauges. Suppose that we work in a gauge which exhibits nonphysical degrees of freedom. When we take the trace over all states in equation (1), we include also these nonphysical degrees of freedom. This is, of course, incorrect, because the nonphysical particles are not in equilibrium with the physical heat bath.

The correct way to avoid this problem is to define the partition function by equation (1), when the right-hand side (r.h.s.) is evaluated in some "physical" gauge that does not contain nonphysical states, or take the trace only over the physical states. To do this, we follow Gross, Pisarski, and Yaffe (1981) and use the $A_0 = 0$ gauge, where the Hamiltonian is given by

$$\mathcal{H} = \tfrac{1}{2}\int d^3x\left[\bar{E}^2 + \tfrac{1}{2}F^2_{ij} + \sum_f \bar{\psi}_f(-i\gamma^i D_i + m_f)\psi_f\right] \tag{11}$$

Here $E^a_i = F^a_{i0}$ is the canonical momentum conjugated to A^a_i.

Our state vector space spanned by the vector set $\{|A^a_i(x)\rangle|\psi_F(x)\rangle\}$ still contains nonphysical states because the gauge is not completely fixed by the condition $A^a_0(x) = 0$. Time-independent gauge transformations are still allowed. This is reflected by the fact that the Gauss's law

$$G^a \equiv \nabla\cdot\bar{E}^a + g f^{abc}\bar{A}^b\bar{E}^c + i\bar{\psi}\gamma^0\tau^a\psi = 0 \tag{12}$$

does not appear in the equations of motion that can be derived from the Lagrangian (7), or from the Hamiltonian (11) by using the canonical commutation relations. The connection between these two statements can be understood by observing that the Gauss' law operator G^a generates time-dependent gauge transformations. Because the physical states must be identified with the states invariant under time-independent gauge transformation, the definition of the physical states will be

$$G^a|\psi_{\text{phys}}\rangle = 0 \tag{13}$$

Thus we should insert the projection operator that projects onto the subspace of physical states (Callan, Dashen, and Gross, 1978)

$$p = \int \mathscr{D}\Lambda(\bar{x}) \exp\left[i \int d^3x \Lambda^a G^a\right] \tag{14}$$

In equation (13), the integration is restricted over the functions $\Lambda^a(\bar{x})$ that vanish sufficiently fast at infinity.

Now the trace in equation (1) can be written as the functional integral over the functions that are periodic in the imaginary time

$$A_i^a(\bar{x}, \beta) = A_i^a(\bar{x}, 0)$$

(antiperiodic for the fermion fields) (Bernard, 1974). Inserting the projection operator from equation (14) into the trace (1) in the following way (Gross, Pisarski, and Yaffe, 1981):

$$Z = \lim_{N\to\infty} \operatorname{Tr}\left\{P \exp\left[-\beta\left(\hat{\mathscr{H}} - \sum_f \mu_f \hat{\mathscr{N}}_f\right)\Big/\right]\right\}^N \tag{15}$$

we get ($it = \tau$)

$$\begin{aligned} Z = \int_{\left(\begin{smallmatrix} A_i \text{ periodic} \\ \psi \text{ antiperiodic}\end{smallmatrix}\right)} & \mathscr{D}\Lambda \mathscr{D}A_i \mathscr{D}E_i \mathscr{D}\bar{\psi}\mathscr{D}\psi \\ & \times \exp\left[-\int_0^\beta d\tau \int d^3x\left(\bar{E}^2 + \frac{1}{2}F_{ij}^2 - i\Lambda^a D_i^{ab} E_i^b - iE_i^a \frac{\partial A_i^a}{\partial \tau}\right.\right. \\ & \left.\left. - \sum_f \bar{\psi}_f \gamma^0 \partial_\tau - i\gamma^i D_i + \gamma^0 \Lambda^a \tau^a + m_f - \mu_f \gamma^0\right)\Psi_f\right] \end{aligned} \tag{16}$$

Note that the boundary conditions of the functions Λ^a at $\tau = 0$ and β are not specified by the construction. As a matter of fact, the partition function (15) is independent of the way we choose these boundary conditions (Montonen). However, for computational simplicity it is convenient to choose also Λ^a to be periodic in the interval $/0, \beta/$. Renaming then Λ^a as

A_0^a and performing the Gaussian integration with respect to E_i^a, equation (16) reads

$$Z \sim \int_{\substack{A_\mu \text{ periodic} \\ \psi \text{ antiperiodic}}} \mathscr{D}A_\mu \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\left[\int_0^\beta d\tau \int d^3x \left(\mathscr{L} + \sum_f \mu_f \bar{\psi}_f \gamma^0 \psi_f\right)\right] \tag{17}$$

The effective action appearing in the exponent of equation (17) is invariant under gauge transformations that preserve the periodicity of the fields. This invariance is a consequence of the periodic boundary conditions we chose for Λ. Had we imposed different boundary conditions, the set of allowed gauge transformations in equation (17) would have been correspondingly different. In order that each unequivalent field configuration will be picked only once, we adopt some gauge condition and apply the standard Faddeev-Popov treatment (Faddeev and Popov, 1967; Lee, 1976). If we choose a linear gauge-fixing equation

$$F^a[A_\mu^b] - c^a(x) = 0 \tag{18}$$

where $c^a(x)$ is a (periodic) c-number function, we obtain

$$Z \sim \int \mathscr{D}A_\mu \mathscr{D}\bar{\psi}\mathscr{D}\psi \delta(F^a - C^a) \det\left(\frac{\partial F}{\partial \omega}\right) \\ \times \exp\left[\int_0^\beta d\tau \int d^3x \left(\mathscr{L} + \sum_f \mu_f \bar{\psi}_f \gamma^0 \psi_f\right)\right] \tag{19}$$

where ω in the Faddeev-Popov determinant, $\det(\partial F/\partial\omega)$, parametrizes the (infinitesimal) gauge transformations. Everything now works similarly as in the zero-temperature field theory. Proceeding in the usual way (Lee, 1976), we eliminate the δ function, obtaining instead a gauge-fixing term in the Lagrangian. Moreover, the Faddeev-Popov determinant can be rewritten as a functional integral over anticommuting (but periodic) Grassman fields. The periodicity of these ghost fields is due to the periodicity of the Faddeev-Popov determinant. (Recall that the Faddeev-Popov determinant represents the variation of the infinitesimal, periodic gauge transformation.) Hence

$$Z \sim \int \mathscr{D}A_\mu \mathscr{D}\bar{\psi}\mathscr{D}\psi D\bar{C}\mathscr{D}C \exp S_{\text{eff}} \tag{20}$$

where the effective action is

$$S_{\text{eff}} = \int_0^\beta d\tau \int d^3x \left[-\frac{1}{4} F_{\mu\nu}^a{}^2 - \frac{1}{2\alpha}(F^a[A_\mu^b])^2 + \bar{C}_a \frac{\partial F^a}{\partial A_\mu^c} D_\mu^{cb} C_b \right. \\ \left. + \sum_f \bar{\psi}_f (i\not{D} + \mu_f \gamma^0 - m_f)\psi_f\right] \tag{21}$$

where ∂_0 should be understood as $\partial_0 = \partial_t = i\partial_\tau$. It is worth noting that the possibility of carrying out the standard Faddeev–Popov treatment here was based on the periodic boundary conditions of the function Λ. Any other choice for the boundary conditions would have led to computational difficulties. Henceforth, we use the covariant gauge (the Lorentz gauge)

$$F^a[A^b_\mu] = -\partial^\mu A^a_\mu \tag{22}$$

Thus, in equation (21) the usual effective Lagrangian for the covariant α gauge (added by the number operator terms) appears:

$$\begin{aligned}\mathscr{L}_{\text{eff}} = -\frac{1}{4} F^a_{\mu\nu}{}^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu_a)^2 + \partial^\mu \bar{C}_a \partial_\mu C_a \\ + g f^{abc} \partial^\mu \bar{C}_a A^b_\mu C^c + \sum_f \bar{\psi}_f (i\not{D} + \mu_f \gamma^0 - m_f)\psi_f\end{aligned} \tag{23}$$

From equation (23), we can easily derive the finite temperature Feynman rules by adding external sources into the partition function and dividing the Lagrangian into a kinetic part and an interaction part: $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$:

$$\begin{aligned}Z(J_\mu, \ldots) \sim \int \mathscr{D}A_\mu \cdots \exp\left\{\int_0^\beta d\tau \int d^3x \, \mathscr{L}_I(A_\mu, \ldots)\right. \\ \left. -\frac{1}{2} A^\mu \left[g_{\mu\nu}\partial^2 - \left(1 - \frac{1}{\alpha}\right)\partial_\mu \partial_\nu \right] A^\nu - J_\mu A^\mu + \cdots \right\}\end{aligned} \tag{24}$$

Here the dots indicate the corresponding terms for fermions and ghosts. The only difference here compared to the vacuum field theory is the finite size of the space in the τ direction. This finiteness of the interval $/0, \beta/$, together with the periodic or (antiperiodic) boundary conditions, lead to the energy Fourier series instead of the Fourier transformation, when we change to the momentum space. We get

$$\begin{aligned}Z[J_\mu, \ldots] &\sim \int \mathscr{D}A_\mu \cdots \exp S_I\left(\frac{-i\delta}{\delta J_\mu}, \ldots\right) \\ &\quad \times \exp\left\{ -\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} A^\mu(n, \bar{k})(k_\mu k_\nu - k^2 g_{\mu\nu}) A^\nu(n, \bar{k}) \right.\right. \\ &\quad \left.\left. + J_\mu(n, \bar{k}) A^\mu(n, \bar{k}) + \cdots \right] \right\} \\ &\sim \exp S_I\left(-i\frac{\delta}{\delta J_\mu}, \ldots\right) \exp\left\{ \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \right. \\ &\quad \left. \times \left[\frac{1}{2} J_\mu(n, \bar{k}) \Delta^{\mu\nu} J_\nu(n, \bar{k}) \right] \right\}\end{aligned} \tag{25}$$

where $\Delta^{\mu\nu}(n, \bar{k})$ is the Feynman propagator, with $k_0 = 2\pi inT$. Thus, the finite temperature Feynman rules are the same as the $T = \mu = 0$ rules with the substitutions

$$\int \frac{d^4k}{(2\pi)^4} \to iT \sum_{k_0} \int \frac{d^3k}{(2\pi)^3}$$

$$\begin{aligned} k_0 &= 2n\pi iT && \text{(for bosons and ghost)} \\ &= (2n+1)\pi iT + \mu && \text{(for fermions)} \end{aligned}$$

$$(2\pi)^4 \delta^{(4)}(k_1 + \cdots + k_N) \to -i\beta(2\pi)^3 \delta_{k_1^0 + \cdots k_N^0} \delta^3(\bar{k}_1 + \cdots \bar{k}_N) \tag{26}$$

2.2. Evaluation of the Frequency Sums

The usual way to perform the frequency sums indicated in equations (26) is to convert them to contour integrals. The method is based on the observation that the function $g_+(z) = (\beta/2)\ \coth(\beta z/2)(g_-(z) = (\beta/2)$ $\tanh[\beta(z-\mu)/2])$ has poles at the points $z = 2\pi niT(z = (2n+1)\pi iT + \mu)$ with residue $+1$. Hence the sum over the even (odd) Matsubara frequencies can be expressed as a contour integral around the imaginary axis, the summand being multiplied by $g_+(z)$ $(g_-(z))$.

To be more specific, let $f(z)$ be a function vanishing sufficiently fast at infinity. If $f(z)$ has no singularities on the imaginary axis, the procedure leads to the equation

$$T \sum_n f(2n\pi iT) = \int_{-i\infty}^{i\infty} \frac{dk_0}{2\pi i} f(k_0) + \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{f(k_0) + f(-k_0)}{e^{\beta k_0} - 1} \frac{dk_0}{2\pi i} \tag{27}$$

When $f(z)$ has no other singularities than simple poles at the complex values $z = \omega_{a'}$ the equation (27) has the useful form

$$\begin{aligned} T \sum_n f(2n\pi iT) = &\int_{-i\infty}^{i\infty} \frac{dk_0}{2\pi i} f(k_0) - \sum_{\operatorname{Re}\omega_a > 0} \frac{1}{e^{\omega_a/T} - 1} \operatorname*{Res}_{k_0 = \omega_a} f(k_0) \\ &+ \sum_{\operatorname{Re}\omega_a < 0} \frac{1}{e^{-\omega_a/T} - 1} \operatorname*{Res}_{k_0 = \omega_a} f(k_0) \end{aligned} \tag{28}$$

For the fermions we get similarly

$$\begin{aligned} T \sum_n f((2n+1)\pi iT + \mu) = &\int_{-i\infty}^{i\infty} \frac{dp_0}{2\pi i} f(p_0) + \oint_C \frac{dp_0}{2\pi i} f(p_0) \\ &- \int_{-i\infty+\mu+\varepsilon}^{i\infty+\mu+\varepsilon} \frac{dp_0}{2\pi i} \frac{f(p_0)}{e^{(p_0-\mu)/T} + 1} \\ &- \int_{-i\infty+\mu-\varepsilon}^{-i\infty+\mu-\varepsilon} \frac{dp_0}{2\pi i} \frac{f(p_0)}{e^{(\mu-p_0)/T} + 1} \end{aligned} \tag{29}$$

Here the contour C runs in the p_0 plane form $\mu - i\infty$ to $\mu + i\infty$ to $0 + i\infty$ to $0 - i\infty$ back to $\mu - i\infty$. Thus the integral over the contour C vanishes when μ goes to zero. On the other hand, the last two terms vanish at zero temperature.

In the case of simple poles we get the simplified form

$$T \sum_n f((2n+1)\pi i T + \mu) = \int_{-i\infty}^{i\infty} \frac{dp_0}{2\pi i} f(p_0) + \sum_{\mathrm{Re}\,\omega_a > 0} \frac{1}{e^{(\omega_a - \mu)/T} + 1} \operatorname*{Res}_{p_0 = \omega_a} f(p_0)$$
$$- \sum_{\mathrm{Re}\,\omega_a < 0} \frac{1}{e^{(-\omega_a + \mu)/T} + 1} \operatorname*{Res}_{p_0 = \omega_a} f(p_0) \tag{30}$$

When the left-hand side (l.h.s.) of equation (30) represents some physical quantity (instead of having just a virtual nature), we can give a transparent interpretation to the terms in the r.h.s. The first is the infinite vacuum fluctuation, the second and third correspond the contribution of the fermions and antifermions, respectively.

3. POLARIZATION TENSOR

3.1. Tensor Structure

In this section we shall study the polarization tensor. We shall first derive the general expressions for nonzero T and μ. The zero-temperature (but finite μ) equations are then obtained as the appropriate limit. We concentrate here on QCD; the QED tensor can then be found as a special case.

The polarization tensor is defined by the Dyson equation

$$\mathcal{D}_{\mu\nu} = \mathcal{D}^0_{\mu\nu} - \mathcal{D}^0_{\mu\alpha} \Pi^{\alpha\beta} \mathcal{D}_{\beta\nu} \tag{31}$$

where $\mathcal{D}^0_{\mu\nu}$ and $\mathcal{D}_{\mu\nu}$ are the bare and exact gluon propagators, respectively. The vacuum polarization tensor that can be formed from the Lorentz tensors $g_{\mu\nu}$ and $k_\mu k_\nu$ satisfies the constraint

$$k_\mu \Pi^{\mu\nu}(k) = 0 \tag{32}$$

which leads to the equation

$$\Pi^{(\mathrm{vac})}_{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_\mu k_\nu) \Pi^{(\mathrm{vac})}(k^2) \tag{33}$$

At finite temperature, equation (32) is no longer valid. Moreover, owing to the existence of a preferred Lorentz frame, the rest frame of the heat bath, the polarization tensor is no longer restricted to be Lorentz covariant, but only $O(3)$ covariant. So, we can express the polarization tensor by four independent symmetric $O(3)$ tensors that we can choose to be, for example, the following (Gross *et al.*, 1981; Kajantie and Kapusta, 1985; Toimela, 1985)

$$
\begin{aligned}
&A_{00} = A_{0i} = A_{i0} = 0 \\
&A_{ij} = \delta_{ij} - k_i k_j / \bar{k}^2 \\
&B_{\mu\nu} = k_\mu k_\nu / k^2 - g_{\mu\nu} - A_{\mu\nu} \\
&C_{\mu\nu} = \frac{1}{\sqrt{2}|\bar{k}|}[(g_{\mu 0} - k_\mu k_0 / k^2) k_\nu + k_\mu (g_{\nu 0} - k_\nu k_0 / k^2)] \\
&D_{\mu\nu} = k_\mu k_\nu / k^2
\end{aligned} \tag{34}
$$

If we use the definition (31) and the Ward identity

$$k_\mu k_\nu \mathscr{D}^{\mu\nu} = -\alpha = \text{gauge parameter}$$

we can write $\Pi_{\mu\nu}$ in the form (Kajantie and Kapusta, 1985; Toimela, 1985)

$$\Pi_{\mu\nu} = (a - k^2) A_{\mu\nu} + (b - k^2) B_{\mu\nu} + c C_{\mu\nu} + \frac{c^2}{2b} D_{\mu\nu} \tag{35}$$

where a, b, and c are functions of the variables k_0 and $\bar{k}$.

3.2. Polarization Tensor in One-Loop Level at Nonzero Temperature

At the one-loop level (second order in the coupling constant) we divide the polarization tensor as follows:

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}^{(\text{vac})} + \Delta\Pi_{\mu\nu}^{Q} + \Delta\Pi_{\mu\nu}^{G} \tag{36}$$

where the vacuum part, $\Pi_{\mu\nu}^{(\text{vac})}$, is of the form (33). In the massless limit $\Pi_{\mu\nu}^{(\text{vac})}$ is (in the momentum-space-subtraction scheme) (Celmaster and Sivers, 1981)

$$\Pi_{\mu\nu}^{\text{MOM}} = \frac{g^2}{96\pi^2}[(-13 + 3\alpha) N + 4 N_f](k^2 g_{\mu\nu} - k_\mu k_\nu) \ln\left(\frac{-k^2}{M^2}\right) \tag{37}$$

Here and hereafter we shall delete the trivial color index dependence δ^{ab}. In equation (37), $-M^2$ is the Euclidean subtraction point.

The quark loop (Figure 1a) contribution to the matter part of the polarization tensor is

$$
\begin{aligned}
\Delta\Pi_{\mu\nu}^{Q} = -2g^2 \sum_f T \sum_{p_0} \int \frac{d^3 p}{(2\pi)^3} \{ & [2 p_\mu p_\nu + k_\mu p_\nu + k_\nu p_\mu - g_{\mu\nu}(p^2 + kp - m^2)] \\
& / (p^2 - m^2)[(k+p)^2 - m^2] \} - \Pi_{\mu\nu}^{Q(\text{vac})}
\end{aligned} \tag{38}
$$

Here we have already subtracted the vacuum part [that contributes to equation (37)]. The remaining matter part is entirely free of ultraviolet

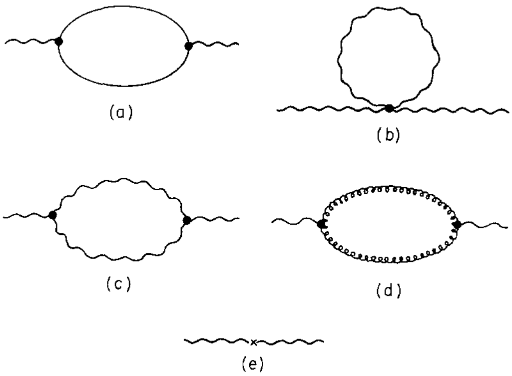

Fig. 1. The polarization tensor at the one-loop level: (a) quark loop, (b, c) gauge loops, (d) ghost loop, (e) counterterm.

divergences and needs no renormalization. The renormalization affects only the vacuum part, equation (37). The corresponding QED tensor is given by equation (38), replacing

$$g^2/2 \to e^2 \tag{39}$$

The last term in equation (36) is a pure non-Abelian contribution, arising from the diagrams (1b–d) and has the form (Kalashnikov and Klimov, 1980a, 1981)

$$\begin{aligned}\Delta\Pi^G_{\mu\nu} = \frac{Ng^2}{2\beta}\sum_{q_0}\int \frac{d^3q}{(2\pi)^3}\{[8q_\mu q_\nu - g_{\mu\nu}(4q^2+8k\cdot q) - 2k_\mu k_\nu + 4q_\mu k_\nu \\ +4k_\mu q_\nu]/q^2(k+q)^2 - 2(\alpha-1)[g_{\mu\nu}(4(k\cdot q)^2 \\ +2q^2k\cdot q - q^2k^2) + 2k^2 q_\mu q_\nu \\ +q^2k_\mu k_\nu - (q^2+3k\cdot q)(k_\mu q_\nu + q_\mu k_\nu)]/q^4(q+k)^2 \\ +(\alpha-1)^2(k_\mu k_\nu(q\cdot k)^2 + q_\mu q_\nu k^4 \\ -k^2k\cdot q(k_\mu q_\nu + q_\mu k_\nu)]/q^4(q+k)^4\} - \Pi^{G(\mathrm{vac})}_{\mu\nu}\end{aligned} \tag{40}$$

In order to get better insight into the structure of the polarization tensor we choose the Feynman gauge ($\alpha = 1$) and perform the summation and the integrations over the angle variables in equations (38) and (40). Setting ($\alpha = 1$) in equation (40) we note immediately that at this one-loop level $\Pi_{\mu\nu}$ satisfies the transversality equation (32). This transversality allows us to write the polarization tensor in the form

$$\Delta\Pi^{\mu\nu}(k) = (k^\mu k^\nu - g^{\mu\nu}k^2)\Delta\Pi^{00}/\bar{k}^2 + \tfrac{1}{2}A^{\mu\nu}(3\Delta\Pi^{00} + \Delta\Pi^\mu_\mu \bar{k}^2/k^2) \tag{41}$$

For the functions $\Delta\Pi^{00}$ and $\Delta\Pi^\mu_\mu$ we get the following expressions (Kapusta, 1979b):

$$\Delta\Pi^{00}=\frac{g^2}{2\pi^2}\sum_f\int_0^\infty\frac{dpp^2}{E_p}n_p\left\{1+\frac{4E_p^2-\omega^2-(2\pi nT)^2}{8p\omega}\right.$$
$$\times\ln\frac{[(2\pi nT)^2+\omega^2+2p\omega]^2+4E_p^2(2\pi nT)^2}{((2\pi nT)^2+\omega^2-2p\omega)^2+4E_p^2(2\pi nT)^2}$$
$$\left.-\frac{2\pi nTE_p}{p\omega}\arctan\frac{16\pi nTp\omega E_p}{[(2\pi nT)^2+\omega^2]^2+4E_p^2(2\pi nT)^2-4p^2\omega^2}\right\}$$
$$+\frac{Ng^2}{\pi^2}\int_0^\infty dqqN_q\left[1+\frac{4g^2-(2\pi nT)^2-2\omega^2}{8q\omega}\right.$$
$$\times\ln\frac{[(2\pi nT)^2+\omega^2+2q\omega]^2+4q^2(2\pi nT)^2}{[(2\pi nT)^2+\omega^2-2q\omega]^2+4q^2(2\pi nT)^2}$$
$$-\frac{2\pi nT}{\omega}\arctan\frac{16\pi nT\omega q^2}{[(2\pi nT)^2+\omega^2]^2+4q^2(2\pi nT)^2-4q^2\omega^2} \tag{42}$$

$$\Delta\Pi^\mu_\mu=\frac{g^2}{\pi^2}\sum_f\int_0^\infty\frac{dpp^2}{E_p}n_p\left[1+\frac{2m_f^2-\omega^2-(2\pi nT)^2}{4p\omega}\right.$$
$$\left.\times\ln\frac{[(2\pi nT)^2+\omega^2+2p\omega]^2+4E_p^2(2\pi nT)^2}{[(2\pi nT)^2+\omega^2-2p\omega]^2+4E_p^2(2\pi nT)^2}\right]$$
$$+\frac{Ng^2}{\pi^2}\int_0^\infty dq\,qNq\left[2-\frac{5[(2\pi nT)^2+\omega^2]}{8q\omega}\right.$$
$$\left.\times\ln\frac{[(2\pi nT)^2+\omega^2+2q\omega]^2+4q^2(2\pi nT)^2}{[(2\pi nT)^2+\omega^2-2q\omega]^2+4q^2(2\pi nT)^2}\right] \tag{43}$$

Here

$$E_p=\sqrt{(p^2+m_f^2)}^{1/2},\qquad n=k_0/2\pi iT,\qquad \omega=|\bar{k}|$$
$$n_p=n_p^++n_p^-=\frac{1}{e^{\beta(E_p-\mu)}+1}+\frac{1}{e^{\beta(E_p+\mu)}+1}$$
$$N_q=\frac{1}{e^{\beta q}-1}$$

Note that here the polarization tensor is given only at points $k_0=2\pi inT$, $n\in Z$. The formulas (42), (43) should be continued analytically if one needs $\Pi_{\mu\nu}$ in the whole complex plane. This is the case if one wants to study the plasma oscillations, for example. At the arbitrary complex values of k_0 the polarization tensor acquires an imaginary part, unlike at the pure imaginary values of k_0, where $\Pi_{\mu\nu}$ is purely real.

The most interesting limit here is the static ($k_0 = 0$) infrared domain. From equation (42) we get

$$\Delta\Pi_{00}(0, \omega) = g^2\left[NT^2/3 + \sum_f \int_0^\infty \frac{dp}{2\pi^2} n_p(p^2 + E_p^2)/E_p\right]$$
$$- Ng^2 T\omega/4 + O(\omega^2 \ln \omega/T) \qquad (44)$$

As a matter of fact these first two terms are independent of the gauge parameter. One can obtain them from the general formulas, equations (38) and (40), and find that the gauge-parameter drops out. The reader could ask whether these terms are really gauge invariant or independent of the gauge in this particular gauge class only. The answer is not known. However, the latter possibility seems to be unlikely. As we shall see in the next section, these two terms are related (at least in the covariant gauge, we use) to the orders of $O(g^3)$ and $O(g^4 \ln g)$ in the perturbation expansion of the thermodynamic potential. These orders should be gauge invariant because the thermodynamic potential is a physical quantity and its gauge dependence which is compensated for by the gauge dependence of the coupling constant does not appear until the order $O(g^4)$. Of course, it is possible that there exists a pathological gauge in which Π_{00} has a different static behavior at long wavelengths than it has in the covariant gauges. However, in this case the orders $O(g^3)$ and $O(g^4/ng)$ of the thermodynamic potential should be generated in a different way than in the covariant gauges, in which they appear as a summation of the infrared behavior of the ring diagrams (see the next section). Hence, the existence of such a pathological gauge is not out of question, but is rather hard to imagine. In order to see what is the gauge dependence of the next term ($\sim\omega^2 \ln \omega/T$), we use equations (38) and (40) instead of equation (42). We consider for simplicity only the transparent massless limit, obtaining

$$\Delta\Pi^{00}_{(0,\omega)} = Ng^2T^2\left(2N + N_f + 3\sum_f \mu_f^2\Big/T^2\pi^2\right)\Big/6 - Ng^2T\omega/4$$
$$+ \frac{g^2}{96\pi^2}[(-13N + 3\alpha)N + 4N_f]\omega^2 \ln\frac{\omega}{T} + O(\omega^2) \qquad (45)$$

Thus, the $\sim\omega^2 \ln \omega/T$ term (and all the higher terms ω) depends on the gauge.

Furthermore, the spatial part $\Delta\Pi_{ij}$ has the form (in the massless limit)

$$\Delta\Pi_{ij}(0, \omega) = -(\delta_{ij} - k_ik_j/\bar{k}^2)\left\{\frac{Ng^2}{64}(\alpha^2 + 2\alpha + g)\omega T\right.$$
$$\left. + \frac{g^2}{96\pi^2}[(13 - 3\alpha)N + 4N_f]\omega^2 \ln\frac{\omega^2}{T^2}\right\} + O(\omega^2) \qquad (46)$$

where even the $\sim\omega T$ is gauge-parameter dependent.

Taking equations (46), (47) into account we can write at $k_0=0$, $\omega\to 0$

$$-i\mathscr{D}_{\mu\nu}(k_0=0,\,\omega\to 0)\simeq\frac{\delta_{\mu 0}\delta_{\nu 0}}{\omega^2+\Delta\Pi_{00}(0,0)}+\frac{\delta_{\mu i}\delta_{\nu j}}{\omega^2} \tag{47}$$

So, we can interpret equation (47) by saying that the thermal fluctuations have generated an electric mass $m_{\rm el}^2=\Delta\Pi_{00}(0,\bar{0})$ for the static electric fields, but the static magnetic fields are unscreened at the one-loop level. However, we must be very careful in our interpretation because the polarization tensor is a gauge-dependent quantity and its static infrared limit is gauge invariant at most up to the lowest order (see the discussion in Section 4).

Note the similar structure of the logarithmic terms in equations (45), (46) compared with equation (37). Hence, if we add the vacuum part $\Pi_{\rm vac}$ to $\Delta\Pi$, we find that the term proportional to $\omega^2\ln\omega$ disappears in $\Pi_{\mu\nu}(0,\omega)$. A similar cancellation has been reported by D. Gross *et al.* (1981) in the limit $k_0\neq 0$, $\omega\to 0$. As a matter of fact, the term $\sim k^2\ln k^2$ in the vacuum polarization tensor will be canceled at every value of k_0 and $\bar{k}$, by the similar term in the matter part. This can be seen by examining how these logarithmic terms arise from the frequency sums. For this, let us consider some arbitrary sum in equation (31). We can write it as

$$T\sum_{q_0}\frac{f(q_0,\bar{q},k)}{q^2}=-\frac{1}{|\bar{q}|}\left(\frac{1}{2}+\frac{1}{e^{|\bar{q}|/T}-1}\right)f(q_0=|\bar{q}|,\bar{q},k)+\cdots \tag{48}$$

where the dots indicate terms arising from the poles of the function $f(q_0)$. The vacuum part comes from the constant term in the brackets after integrating over the $\bar{q}$ space. The Bose-Einstein distribution in the brackets gives the matter part and its logarithmic terms arises from the $\bar{q}$-independent term in the expansion

$$\frac{1}{e^{q/T}-1}=\frac{T}{q}-\frac{1}{2}+O(q) \tag{49}$$

when integrated over the small $\bar{q}$ domain. This term $k^2\ln k^2/T^2$ added to the vacuum polarization cancels the logarithmic dependence on k^2 there, leaving a term $\sim k^2\ln T^2/-M^2$.

What is most important in equations (45), (46) is the appearance of the linear terms. Such linear terms do not exist in QED, because they arise from the gluon and ghost loops (Figure 1c–d). Nevertheless, this phenomenon is not associated with the non-Abelian character of QCD, but it is rather a bosonic behavior. It occurs also in scalar electrodynamics (Kalashnikov and Klimov, 1980b; Toimela, 1983a).

3.3. Polarization Tensor at the Zero Temperature

We shall now discuss the zero temperature limit of the polarization tensor. In Section 3.2 we gave the expression for $\Pi_{\mu\nu}(q_0, \bar{q})$ at the set of the imaginary values $q_0 = 2\pi i n T$, where $n \in Z$. At the zero temperature limit we shall need its expression for $q_0 = i\tilde{q}_0$, where $\tilde{q}_0$ is real and continuous. This analytic continuation from the above-mentioned distinct points to the whole imaginary axis is not unique, but becomes uniquely defined, when it is required that $\Pi_{\mu\nu}$ do not have an essential singularity at $q_0 = \infty$.

We find it more convenient to use here the Euclidean metric instead of the Minkowski metric. The Euclidean polar coordinate q, ϕ, θ, φ are defined through the equations

$$
\begin{aligned}
q &= (\tilde{q}_0^2 + \bar{q}^2)^{1/2} \equiv (-q_0^2 + \bar{q}^2)^{1/2} \\
\phi &= \arctan(|\bar{q}|/\tilde{q}_0)
\end{aligned}
\tag{50}
$$

θ and φ are the angle variables of the original three-space. By using these polar coordinates we write the zero temperature limit of equations (42), (43) in the form

$$
\begin{aligned}
\Delta\Pi^{00}(q, \phi, \mu) = \frac{g^2}{2\pi^2} \sum_f \int_0^{(\mu_f^2 - m_f^2)^{1/2}} \frac{dp}{E_p} p^2 \Bigg[1 + \frac{4E_p^2 - q^2}{8pq \sin\phi} \\
\times \ln \frac{(q + 2p\sin\phi)^2 + 4E_p^2 \cos^2\phi}{(q - 2p\sin\phi)^2 + 4E_p^2\cos^2\phi} \\
- \frac{E_p}{p} \cot\phi \arctan \frac{8pE_p \sin\phi\cos\phi}{q^2 + 4E_p^2\cos^2\phi - 4p^2\sin^2\phi} \Bigg]
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\Delta\Pi^{\mu}_{\mu}(q, \phi, \mu) = \frac{g^2}{\pi^2} \sum_f \int_0^{(\mu_f^2 - m_f^2)^{1/2}} \frac{dp}{E_p} p^2 \Bigg[1 + \frac{2m_f^2 - q^2}{8pq\sin\phi} \\
\times \ln \frac{(q + 2p\sin\phi)^2 + 4E_p^2\cos^2\phi}{(q - 2p\sin\phi)^2 + 4E_p^2\cos^2\phi} \Bigg]
\end{aligned}
\tag{52}
$$

These equations are, of course, gauge invariant.

The integrals of equations (51), (52) can be done in a closed form (see Appendix A for the details). The resulting formulas are

$$
\begin{aligned}
\Delta\Pi_{00} = \frac{g^2}{\pi^2} \sum_f \Bigg\{ \frac{2}{3} \mu(\mu^2 - m^2)^{1/2} - \frac{q^2 \sin^2\phi}{6} \ln \frac{\mu + (\mu^2 - m^2)^{1/2}}{m} \\
+ \frac{4\mu^3 - 3q^2\mu}{24 q \sin\phi} \ln \frac{4\mu^2\cos^2\phi + [2(\mu^2 - m^2)^{1/2}\sin\phi + q]^2}{4\mu^2\cos^2\phi + [2(\mu^2 - m^2)^{1/2}\sin\phi - q]^2}
\end{aligned}
$$

$$+\frac{(2m^2-q^2)(q^2+4m^2)^{1/2}}{24q}\sin^2\phi$$
$$\times\ln\frac{2\mu^2(2m^2+q^2)-2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}{2\mu^2(2m^2+q^2)+2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}$$
$$-\frac{1}{2}(\mu^2-\frac{1+2\sin^2\phi}{12}q^2)\cot\phi$$
$$\times\arctan\left[\frac{\mu(\mu^2-m^2)^{1/2}\sin 2\phi}{\mu^2\cos 2\phi+m^2\sin^2\phi+q^2/4}\right]\Bigg\} \tag{53}$$

$$\Delta\Pi^{\mu}_{\mu}=\frac{g^2}{2\pi^2}\sum_f\Bigg(\mu(\mu^2-m^2)^{1/2}-\frac{q^2}{2}\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m}$$
$$+\frac{2m^2-q^2}{4q}\Bigg\{\frac{\mu}{\sin\phi}\ln\frac{4\mu^2\cos^2\phi+(2(\mu^2-m^2)^{1/2}\sin\phi+q)^2}{4\mu^2\cos^2\phi+(2(\mu^2-m^2)^{1/2}\sin\phi-q)^2}+\frac{(q^2+4m^2)^{1/2}}{2}$$
$$\times\ln\frac{2\mu^2(2m^2+q^2)-2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}{2\mu^2(2m^2+q^2)+2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}$$
$$-q\cot\phi\arctan\left[\frac{\mu(\mu^2-m^2)^{1/2}\sin 2\phi}{\mu^2\cos 2\phi+m^2\sin^2\phi+q^2/4}\right]\Bigg\}\Bigg) \tag{54}$$

The ultrarelativistic limits ($m=0$) of these equations agree with Kapusta (1979b)

$$\Delta\Pi_{00}(m=0)=\frac{g^2}{2\pi^2}\sum_f\left[\frac{2}{3}\mu^2+\frac{\mu(4\mu^2-3q^2)}{24q\sin\phi}\ln\frac{\cos^2\phi+(\sin\phi+q/2\mu)^2}{\cos^2\phi+(\sin\phi-q/2\mu)^2}\right.$$
$$-\frac{q^2\sin^2\phi}{24}\ln\left(1+\frac{8\mu^2\cos 2\phi}{q^2}+\frac{16\mu^4}{q^4}\right)$$
$$\left.-\frac{1}{2}\left(\mu^2-\frac{1+2\sin^2\phi}{12}q^2\right)\cot\phi\arctan\left(\frac{\sin 2\phi}{\cos 2\phi+q^2/4\mu^2}\right)\right] \tag{55}$$

$$\Delta\Pi^{\mu}_{\mu}(m=0)=\frac{g^2}{2\pi^2}\sum_f\left[\mu^2-\frac{q\mu}{\sin\phi}\ln\frac{\cos^2\phi+(\sin\phi+q/2\mu)^2}{\cos^2\phi+(\sin\phi-q/2\mu)^2}\right.$$
$$-\frac{q^2}{8}\ln\left(1+\frac{8\mu^2\cos 2\phi}{q^2}+\frac{16\mu^4}{q^4}\right)$$
$$\left.+\frac{q^2\cot\phi}{4}\arctan\left(\frac{\sin 2\phi}{\cos 2\phi+q^2/4}\right)\right] \tag{56}$$

In Section 5 we shall need the zero momentum limit of the polarization tensor. In this limit equations (53), (54) reduce to

$$\Delta\Pi_{00}(0,\phi)=\frac{g^2}{2\pi^2}\sum_f\left\{\mu(\mu^2-m^2)^{1/2}-\mu^2\cot\phi\arctan\left[\frac{(\mu^2-m^2)^{1/2}}{\mu\cot\phi}\right]\right\} \tag{57}$$

$$\Delta\Pi^{\mu}_{\mu}(0,\phi)=\frac{g^2}{2\pi^2}\sum_f\left\{\mu(\mu^2-m^2)^{1/2}-m^2\cot\phi\arctan\left[\frac{(\mu^2-m^2)^{1/2}}{\mu\cot\phi}\right]\right\} \tag{58}$$

Turning to QED by the substitution (39), we find the above results, equations (57), (58), to be in agreement with those of Akhiezer and Peletminsky (1960), after the correction of a typographic error in their paper.

4. THERMODYNAMIC POTENTIAL

4.1. Ideal Gas of Quarks and Gluons

We shall now compute the thermodynamic potential in the perturbation theory up to the few lowest orders. The calculation can be carried out by using the definition (8) and the formulas (19)-(22). The ideal gas limit is obtained by putting $g=0$ and then performing the Gaussian integrations in equation (20). The integrations give

$$\Omega_0=-\frac{T}{V}\ln\left[\det{}^{-1/2}(\partial^2 g^{\mu\nu}\delta^{ab})\det(\partial^2\delta^{ab})\prod_f\det(i\partial\!\!\!/_f-m_f)\right], \tag{59}$$

where the three determinants originate from gauge, ghost, and fermion fields, respectively. The gauge and ghost determinants should be evaluated on the space of periodic functions, while the fermion determinants on the space of antiperiodic functions, as discussed in Section 2. In the fermion determinant, ∂_f should be understood as

$$\partial\!\!\!/_f=i\gamma^0(\partial_\tau-\mu_f)+\bar\gamma\cdot\nabla \tag{60}$$

It is worth noting that also in QED we have the ghost determinant, if the covariant gauge is used. It is necessary to take this ghost determinant into account in order to get the familiar black-body radiation formula with the right coefficient (Bernard, 1974; Shuryak, 1980). However, the ghost does not have interactions in QED, hence there the ideal gas formula (59) is the only place where the ghost appears to have any contribution.

The determinants in equation (59) can be evaluated in the momentum space using the technique described in Section 2.2. After subtracting the vacuum contribution we get

$$\Omega_0 = -\frac{\pi^2}{45}(N^2-1)T^4 - \frac{N}{3\pi^2}\sum_f \int_0^\infty \frac{dpp^4}{E_p}\left(\frac{1}{e^{(E_p-\mu_f)/T}+1} + \frac{1}{e^{(E_p+\mu_f)/T}+1}\right) \quad (61)$$

This equation could have been written down immediately without using the whole machinery developed in Section 2. However, we consider it useful to demonstrate how the ideal gas formula emerges from the theory. Furthermore, the way we have derived equation (61) illustrates the role of the ghost and shows how they cancel out the contribution of the superfluous degrees of freedom appearing in the gauge propagator.

In the ultrarelativistic limit ($m_f = 0$), the fermion integral results in a fourth-order homogeneous polynomial of the variables T and μ_f:

$$\Omega_0(m=0) = -\frac{\pi^2}{45} T^4 \left\{ N^2 - 1 + \frac{7}{4} NN_f + 15N \sum_f \left[\mu_f^2/2\pi^2 T^2 + \mu_f^4/(2\pi^2 T^2)^2 \right] \right\} \quad (62)$$

On the other hand, at low density $\mu_f' = \mu_f - m_f \ll m_f$, T, we can use the classical statistics approximation replacing the Fermi-Dirac distribution by the Boltzmann one. The integral can then be expressed by the Bessel function of the second order, $K_2(z)$ in the following way:

$$\Omega_0 = -\frac{N}{\pi^2} T^2 \sum_f m_f^2 K_2(m_f/T) e^{\mu_f'/T} - \frac{\pi^2}{45} T^4 (N^2-1) \quad (63)$$

The nonrelativistic limit ($T \ll m_F$) of this formula is

$$\Omega_0 = -\frac{2N}{(2\pi)^{3/2}} T^{5/2} \sum_f m_f^{3/2} e^{\mu_f'/T} \quad (64)$$

4.2. Beyond the Ideal Gas Approximation: The Exchange Energy

For calculating perturbative correction to the ideal gas formula (61) we use the old trick of differentiating Ω with respect to the coupling constant. Hence, we have the equation

$$\Omega(g) = \Omega(0) + \int_0^g dg' \frac{\partial \Omega}{\partial g'} \quad (65)$$

The derivative $\partial\Omega/\partial g$ can be expressed in the form (Kalashnikov and Klimov, 1981a; Kalashnikov, 1984)

$$\frac{\partial\Omega}{\partial g}=\frac{-i}{2g}\frac{i}{\beta}\sum_{k_0}\int\frac{d^3k}{(2\pi)^3}\Bigg[\Pi^{\mu\nu}_{ab}(k)\mathscr{D}^{ab}_{\mu\nu}(k)-\sum_f S_f(k)\,\Sigma_f(k)$$
$$-G(k)\,\Sigma_G(k)+\frac{1}{6\beta}\sum_{q_0}\int\frac{d^3q}{(2\pi)^3}\Gamma^{abc}_{0\mu\nu\lambda}(k,q,-k-q)\mathscr{D}^{\mu\mu'}_{aa'}(k)$$
$$\times\mathscr{D}^{\nu\nu'}_{bb'}(q)\mathscr{D}^{\lambda\lambda'}_{cc'}(k+q)\Gamma^{a'b'c'}_{\mu'\nu'\lambda'}(-k,-q,k+q)\Bigg] \tag{66}$$

Here (k), $S_F(k)$, and $G(k)$ are the exact gauge, fermion, and ghost propagators; Π is the polarization tensor discussed in the preceding section, and Σ_G and Σ_f are the (one-particle irreducible) self-energies of ghost and fermions, respectively. Γ_0 and Γ are the bare and exact three-point functions. The sum over k_0 should, of course, in the second (fermion) term count the odd Matsubara frequencies, while in other terms it runs over the even frequencies. Equation (66) is expressed graphically in Figure 2. In QED we have only the two first terms in equation (66), which moreover appear to be equal there. Hence the correspondent QED formula is rather simple.

The first perturbative correction (of the order g^2) is found by replacing the exact propagators and the vertex function Γ by the bare ones and taking into account the lowest-order diagrams in the self-energies. The contributing diagrams are shown in Figure 3.

When we calculate these diagrams, we encounter ultraviolet divergencies. However, because only the energy differences are measurable, we subtract the infinite T, μ-independent vacuum contribution from these diagrams. After this the T, μ-dependent infinities arising from diagrams (3a–d) cancel against the infinite counterterm diagrams (3e–g). What

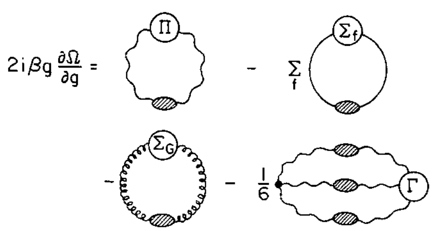

Fig. 2. Graphical representation of equation (66), The wavy, solid, and dotted lines with a shaded "bubble," correspond to exact gauge, quark, and ghost propagators, respectively.

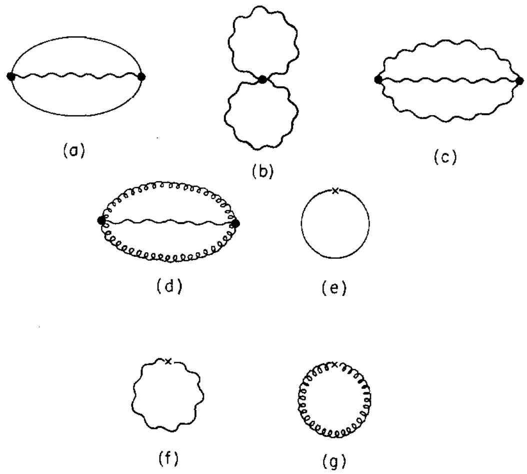

Fig. 3. $O(g^2)$ contributions to the thermodynamic potential.

remains is the finite part of the diagrams (3a–d), that is (Kapusta, 1979a),

$$
\begin{aligned}
\Omega_{(\mathrm{exch})} = {} & \frac{g^2 N(N^2-1)}{144} T^4 + \frac{(N^2-1) g^2 T^2}{24\pi^2} \sum_f \int_0^\infty \frac{dp}{E_p} p^2 n_p \\
& + \frac{(N^2-1) g^2}{32\pi^5} \sum_f \int_0^\infty \frac{dp\,dq}{E_p E_q} p^2 q^2 \left[\left(2 + \frac{m_f^2}{pq} \ln \frac{E_p E_q - m_f^2 - pq}{E_p E_q - m_f^2 + pq} \right) \right. \\
& \times (n_p^- n_q^- + n_p^+ n_q^+) \\
& \left. + \left(2 + \frac{m_f^2}{pq} \ln \frac{E_p E_q + m_f^2 + pq}{E_p E_q + m_f^2 - pq} \right) (n_p^- n_q^+ + n_q^- n_p^-) \right]
\end{aligned}
\tag{67}
$$

We refer to this term as the exchange correction, because its zero temperature limit is the exchange energy corresponding to the process where fermions change their places in the Fermi sphere.

Here it is worth mentioning something about our way of performing the frequency sums discussed in Section 2. We have used the analytic continuation in order to calculate the sums by the contour integral method. However, because the original frequency sum was over a set of distinct points having no convergence point in the finite complex plane, the analytic continuation is not unique. Consequently, as found already by Kapusta (1979a), the result will depend on what order we perform our sums (more precisely: it will depend on what sum is eliminated by the energy conserving Kronecker delta associated with the vertices). The way to avoid this problem is to replace the Kronecker delta by its integral representation as discussed by Norton and Cornwall (1975) and by Kapusta (1979a), if the sums are performed via equations (27) and (29). Let us stress here that all the ambiguities arise because of the nonunique analytic continuation. It is entirely possible to obtain the right result by direct summation (Kapusta 1979a), i.e., using the contour integral method but doing the frequency sums one by one, by deforming the contour of first integral to infinity according to equation (28) *before* the second summation will be done. However, this method will be more laborious than the method used by Norton and Cornwall and by Kapusta.

The massless limit of equation (67) can again be obtained exactly:

$$\Omega_{\text{exch}}(m=0)=\frac{N^2-1}{144}\,g^2T^4\left\{N+\frac{5}{4}\,N_f+g\sum_f\left[\mu_f^2/2\pi^2T^2+\mu_f^4/(2\pi^2T^2)^2\right]\right\} \tag{68}$$

whereas the nonrelativistic, low-density limit, $\mu_f' \ll T \ll m_f$ is

$$\Omega_{\text{exch}}=-\frac{N^2-1}{32\pi^4}\,g^2T^2\sum_f m_f^2 e^{2\mu_f'/T} \tag{69}$$

4.3. Correlation Correction

In Section 4.2 we calculated the thermodynamic potential at the two-loop level. If we want to extend our perturbative calculations up to the three-loop level, we encounter infrared singularities. Furthermore, at higher loop levels the infrared divergences become more severe order by order. The appearance of these divergences can be understood through the screening effect. This screening is caused by the effective mass acquired by the timelike gluons at the one loop level [see equation (44)]. In the normal expansion of the perturbation theory the exact gluon propagator in equation (66) will be expanded in terms of the polarization tensor and the bare propagator in the following way:

$$\mathscr{D}=\mathscr{D}^0\sum_{n=0}^{\infty}(-\Pi\mathscr{D}^0)^n \tag{70}$$

In this series, the higher terms become increasingly worse infrared divergent owing to the nonzero value of the timelike polarization tensor at zero momenta.

The way to overcome this problem is already known in nonrelativistic many-body theory (Gell-Mann and Bruckner, 1957; Fetter and Walecka, 1971). We must resum the series in equation (70) (i.e., sum the ring diagrams shown in Figure 4) in order to obtain an infrared convergent expression. Note that we need sum only the contribution of timelike gluons to obtain the dominant contribution of the correlation term (the so-called plasmon

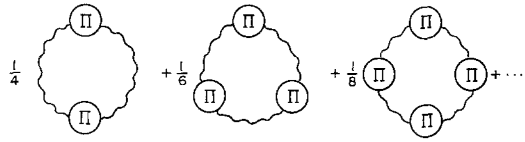

Fig. 4. The ring diagrams contributing to the correlation term.

term):

$$\Omega_{\text{plas}} = \frac{N^2-1}{2\beta} \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \left[\ln\left(1 + \frac{\Pi_{00}}{\bar{k}^2}\right) - \frac{\Pi_{00}}{\bar{k}^2} \right] \tag{71}$$

Here as everywhere else in this section, Π refers to the renormalized polarization tensor.

The contributions of the ring diagrams (Figure 4) corresponding to the spatial gluons can be divided into two parts: the infrared convergent three-loop contribution

$$\Omega_T^{(4)} = -\frac{N^2-1}{2\beta} \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} (\Pi_T^{(2)}/k^2) \tag{72}$$

and the contribution of the remaining, infrared divergent, diagrams

$$\Omega_T^{(\text{rest})} = \frac{N^2-1}{\beta} \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} [\ln(1+\Pi_T/k^2) - \Pi_T/k^2 + (\Pi_T^{(2)})^2/2k^4] \tag{73}$$

$\Pi_T(k_0\bar{k})$ is the combination of the components of the polarization tensor corresponding to the spatial gluons:

$$\Pi_T(k_0, \bar{k}) = \tfrac{1}{2}(\Pi^{\mu}_{\mu}(k_0, \bar{k}) + \Pi_{00}(k_0, \bar{k})k^2/\bar{k}^2) \tag{74}$$

The superscript 2 refers to the order of perturbation in the coupling constant g. The contribution of equations (72) and (73) are of the order $O(g^4)$ and $O(g^6 \ln g)$, respectively. We shall omit them here because we will extract only the contributions up to $O(g^4 \ln g)$. To be consistent we shall also omit the other three-loop diagrams not included in the first diagram of Figure 4. They will all contribute to the order $O(g^4)$. Moreover, we use the expansion

$$\Omega_{\text{plas}} = \frac{N^2-1}{2\beta} \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \Big[\ln(1 + \Delta\Pi_{00}/\bar{k}^2) - \Delta\Pi_{00}/\bar{k}^2$$
$$-(\Pi_{00}^{(\text{vac})})^2/2\bar{k}^4 - \Pi_{00}^{(\text{vac})}\Delta\Pi_{00}/\bar{k}^4 \Big] + O(g^5) \tag{75}$$

The last two terms are infrared convergent and of the order $O(g^4)$ and will hence be ignored. Further, we replace $\Delta\Pi_{00}$ by its second-order counterpart $\Delta\Pi_{00}^{(2)}$, because the higher corrections to $\Delta\Pi$ will give rise to higher corrections to Ω only.

Using equation (71) or (75) we have cured the infrared divergences. However, after ignoring the contribution of $\Pi_{00}^{(\mathrm{vac})}$, the formula (75) is still logarithmically ultraviolet divergent. As we shall see in the next section, the situation at $T=0$ is different. There the correspondent equation (102) is free of ultraviolet divergences. This difference arises from the self-interaction of gluons that contribute at $T\neq 0$ already at the lowest order in perturbation theory. The ultraviolet divergent ring diagram is the first diagram in Figure 4, where both polarization tensor insertions correspond to the graph (1.b). The ultraviolet divergence of this diagram, with the other three-loop diagrams, not included in Figure 4, cancel the divergence of the counterterm diagram associated with the vertex corrections of Figure 3b. However, we shall not go into the renormalization of the fourth-order diagrams, because it will affect only the term of the order $O(g^4)$. Instead we shall concentrate on the infrared structure of the diagrams, because this infrared structure will give the nonanalytic terms in g^2.

Using equation (42) we find that

$$\Delta\Pi_{00}^{(2)}(n,\bar{0})=0, \qquad n\neq 0 \tag{76}$$

Thus, when $n\neq 0$, we can expand the logarith in equation (71) up to the fourth order. Hence, the only term that gives a contribution below the order $O(g^4)$ is the $n=0$ term. We divide this term into two parts:

$$\Omega_1^{\mathrm{plas}}=\frac{N^2-1}{2\beta}\int\frac{d^3k}{(2\pi)^3}\left[\ln\left(1+\frac{\Lambda(0)}{\omega^2}\right)-\frac{\Lambda(0)}{\omega^2}\right] \tag{77}$$

and

$$\Omega_2^{\mathrm{plas}}=\frac{N^2-1}{2\beta}\int\frac{d^3k}{(2\pi)^3}\left(\ln\{[1+\Lambda(\omega)/\omega^2]/[1+\Lambda(0)/\omega^2]\}-\frac{\Lambda(\omega)-\Lambda(0)}{\omega^2}\right) \tag{78}$$

Here we have used the shorthand notation

$$\Lambda(\omega)=\Delta\Pi_{00}^{(2)}(0,\omega), \qquad \omega=|\bar{k}|$$

The integral in equation (77) is easily integrated, giving (Kapusta, 1979a)

$$\begin{aligned}\Omega_1^{(\mathrm{plas})}&=-\frac{N^2-1}{12\pi}T(\Delta\Pi_{00}^{(2)}(0,\bar{0}))^{3/2}\\&=-\frac{N^2-1}{12\pi}Tg^3\left[\frac{N}{3}T^2+\sum_f\frac{1}{2\pi^2}\int_0^\infty\frac{dp}{E_p}(2p^2+m_f^2)n_p\right]^{3/2}\end{aligned} \tag{79}$$

Ignoring again $O(g^4)$ contributions we write $\Omega_2^{(\text{plas})}$ as

$$\Omega_2^{(\text{plas})} = -\frac{N^2-1}{12\pi^2} T \left\{ \int_0^\infty \frac{\Lambda(\omega)\Lambda'(\omega)}{1+\Lambda(\omega)/\omega^2} \frac{d\omega}{\omega} + 2\int_0^\infty \frac{\Lambda(0)[\Lambda(\omega)-\Lambda(0)]\,d\omega}{\omega^2[1+\Lambda(\omega)/\omega^2][1+\Lambda(0)/\omega^2]} \right\} \tag{80}$$

In the numerators we use the approximation

$$\Lambda(\omega)-\Lambda(0) \simeq \omega\Lambda'(\omega) \simeq \frac{\omega\Lambda'(0)}{1+\omega^2/T^2} \tag{81}$$

and in the other places in equation (80) replace $\Lambda(\omega)$ by $\Lambda(0)$. Note that the possible approximations for the function $\Lambda'(\omega)$ have some arbitrariness. Every sufficiently regular function, having no other g dependence but being proportional to g^2, and having the same behavior at small ω $[\sim\Lambda'(0)]$, and vanishing at large ω, can be used to mimic the function $\Lambda'(\omega)$, and will give the same result [apart a term of the order $O(g^4)$, which we are not interested in]. After the substitution of equation (81) to (80), the integrals can be worked out, exactly, giving

$$\Omega_2^{(\text{plas})} = \frac{N^2-1}{8\pi^2} T\Lambda(0)\Lambda'(0) \ln\frac{\Lambda(0)}{T^2} + O(g^4) \tag{82}$$

By inserting $\Lambda(0) = \Delta\Pi_{00}(0,\bar{0})$ and $\Lambda'(0) = [\partial\Delta\Pi_{00}(0,\omega)/\partial\omega]|_{\omega=0}$ from equation (44) to (82) we get

$$\Omega_2^{(\text{plas})} = -\frac{N(N^2-1)}{32\pi^2} T^2 \left[\frac{N}{3}T^2 + \sum_f \frac{1}{2\pi^2}\int_0^\infty \frac{dp}{E_p}(2p^2+m_f^2)n_p\right] + O(g^4) \tag{83}$$

The contributions of the order $O(g^3)$, equation (79), and $O(g^4 \ln g)$, equation (83), have a simpler form in the zero-mass limit (Toimela, 1983b):

$$\Omega^{(\text{plas})}(m=0) = -\frac{N^2-1}{12\pi} T^4 \left(\frac{N}{3}+\frac{1}{6}N_f+\frac{\Sigma_f \mu_f^2}{2\pi^2T^2}\right)^{3/2} g^3 - \frac{N(N^2-1)}{32\pi^2} T^4 \left(\frac{N}{3}+\frac{1}{6}N_f+\frac{\Sigma_f \mu_f^2}{2\pi^2T^2}\right) g^4 \ln g + O(g^4) \tag{84}$$

It should be noted here that equations (79)-(84), although valid at every $T \neq 0$, do not represent the dominant contribution of the ring diagrams (Figure 4) at low temperatures. The higher terms (in g) are small only when $T \gg \mu_f$. The coefficients of these higher powers (g^n, $n \geq 4$, also $g^n \ln g$, $n \geq 5$) are functions of the ratio T/μ_f and these functions actually diverge when $T/\mu_f \to 0$. Hence, although both corrections (79) and (83) vanish at zero temperature, we cannot conclude that the correlation at $T = 0$ is of the order

$O(g^4)$. As a matter of fact, the divergences in these coefficient functions of the higher powers sum up at $T=0$ giving a contribution of the order $O(g^4 \ln g)$. We shall see this in the next section where we shall calculate this order explicitly.

The logarithmic term in equation (84) does not have any counterpart in the Abelian QED. However, it is worth noting that the thermodynamic potential is not the only place in QCD where that kind of behavior appears. Jackiw and Templeton (1981) have reported similar logarithmic behavior in the three-dimensional field theory. Nevertheless, these two things have different origins. $\Omega_2^{(\mathrm{plas})}$ in the thermodynamic potential is related via Π_{00} to the electric fields. On the other hand, the three-dimensional (Euclidean) QCD corresponds to the high-temperature behavior of the magnetic fields at the length scale $\xi \gtrsim (g^2 T)^{-1}$, where the electric fields are decoupled due to the electric mass (Appelquist, 1981).

Comparing equation (84) with (68) we see that the correlation terms overwhelm the exchange term, unless g^2 is extremely small. This paradox can be explained by taking into account the "modified plasmon" (Källman and Toimela, 1983). Both terms of equation (84) are related to the polarization tensor by equations (79) and (82). The limit $\Delta\Pi_{00}(0, \bar{0})$ appearing in those equations is calculated up to the third order $\sim g^3$ (Kajantie and Kapusta, 1982) resulting in

$$\Delta\Pi_{00}(0,\bar{0}) = \Delta\Pi_{00}^{(2)}(0,\bar{0}) - \frac{5}{16\pi} N g^2 T [\Delta\Pi_{00}^{(2)}(0,\bar{0})]^{1/2} \tag{85}$$

The substitution of equation (85) into (79) and (82) represents the "modified plasmon." Equation (84) then reads

$$\begin{aligned}\tilde{\Omega}^{(\mathrm{plas})} = &-(N^2-1)T[\Delta\Pi_{00}^{(2)}(0,\bar{0})]^{3/2}/12\pi \\ &-(N^2-1)T^2/32\pi^2 \Delta\Pi_{00}^{(2)}(0,\bar{0}) \ln(\Delta\Pi_{00}^{(2)}(0,\bar{0})/T^2)\end{aligned} \tag{86}$$

where in the small fermion mass limit

$$\begin{aligned}\Delta\Pi_{00}(0,\bar{0}) = &\left(2N + N_f + 3\sum_f \mu_f^2/\pi^2 T^2\right) g^2 T^2/6 \\ &-5N\left(2N + N_f + 3\sum_f \mu_f^2/\pi^2 T^2\right)^{1/2} g^3 T^2/16\sqrt{6}\pi\end{aligned} \tag{87}$$

The modified plasmon terms could be expanded by powers of g^2, and they differ from the naive plasmon for the first time in the order $O(g^4)$. The reader may wonder why this higher-order contribution has been taken into account, because we have not calculated the other $O(g^4)$ terms, and especially because there exist two ambiguities with this modified plasmon.

Firstly, it has been evaluated in the temporal axial gauge, where the fields $A_i(x)$ are not periodic in the imaginary time, and thus the true Feynman rules are not known. Secondly, the modification for the plasmon terms as defined through the $O(g^3)$ correction in $\Pi_{00}(0, \bar{0})$ is gauge dependent (Toimela, 1985). [Equation (85) is valid only in the temporal axial gauge.]

We shall answer these questions here. Let us first consider the problems with the temporal axial gauge at finite temperature. It is true that for a given periodic $A_\mu(x)$ in the action of equation (16) there is not necessarily any periodic gauge transformation that leads to a potential having $A_0^a(x) = 0$. If we anyway, use the $A_0 = 0$ gauge, the fields $A_i(x)$ are no longer periodic and we expect to obtain Feynman rules that are the usual finite temperature Feynman rules plus some corrections. It is not known in which order of the perturbation theory these corrections appear for the first time. However, by ignoring these unknown corrections some gauge-invariant quantities have been calculated in the $A_0 = 0$ gauge yielding the same results as in the other gauges. In this way, the plasmon term of the thermodynamic potential has been evaluated up to $O(g^3)$ in the $A_0 = 0$ gauge (Källman and Toimela, 1983) and the result agrees with equation (79) here. The $O(g^4 \ln g)$ term has not been calculated in the temporal gauge, but it is immediately clear by virtue of connected results (Källman and Toimela, 1983; Toimela, 1983b) that the same result as our equation (83) will arise also in the $A_0 = 0$ gauge. Those plasmon terms mentioned above are interesting in this context because they are obtained by an analogous summation of the infrared divergent diagrams as should be done for obtaining the $O(g^3)$ correction to $\Pi_{00}(0, \bar{0})$. Hence it seems that whatever the corrections to the periodic rules in the $A_0 = 0$ gauge are, they do not affect the dominant infrared structure. Thus, we expect that $\Pi_{00}(0, \bar{0})$ up to $O(g^3)$ is not affected by the nonperiodicity corrections of the Feynman rules in the $A_0 = 0$ gauge.

The "modified plasmon," if defined through the $O(g^3)$ correction in $\Pi_{00}(0, \bar{0})$, is gauge dependent, but so are also all the terms of the order $O(g^4)$ or higher, owing to the gauge dependence of the coupling constant. Hence we must accept the gauge dependence of the $O(g^4)$ term (as a part of the renormalization prescription). One can ask what gauge choice will give the best convergence of the perturbation expansion (i.e., in which gauge the higher order corrections will be small). It has been argued (Celmaster and Sivers, 1981) that the axial gauge will be the best choice because it has no spurious degrees of freedom. However, after choosing the axial gauge

$$n_\mu A^\mu(x) = 0$$

we must further specify the vector n_μ. We shall argue that the best choice here will be the temporal axial gauge

$$n_\mu = (1, \bar{0})$$

In this gauge the polarization tensor is related to the electric screening mass by the equation

$$m_{\mathrm{el}}^2 = \Pi_{00}(0, \bar{0}) \tag{88}$$

This equation is not valid in an arbitrary gauge. The fact that equation (88) is true at every order of the perturbation theory in the temporal gauge, is based on two facts. Firstly, of course, the temporal gauge admits a Hamiltonian formulation and thus the linear response analysis of the screening can be carried out. Secondly, the fact which makes the temporal gauge different from all the other gauges that exhibit Hamiltonian formulation (e.g., Coulomb gauge) is that the electric field is linear in the potential field:

$$E_i^a = -\partial_0 A_i^a$$

and hence equation (88) will not acquire any higher-order corrections from the nonlinear terms (as will be the case in the Coulomb gauge, for example). Hence, because the plasmon is physically based on the screening and because the modified plasmon, calculated in the temporal gauge, takes the screening into account up to a higher order, we argue that the modified plasmon, defined by equation (85), represents the dominant contribution beyond the order $O(g^4 \ln g)$ and hence its inclusion will be relevant.

In QED we do not have a term of the order $O(e^4 \ln e)$, as mentioned above, owing to the different infrared structure of $\Pi_{00}(0, \omega)$ (i.e., the absence of a linear term in ω). Thus the correlation in QED is given simply by

$$\Omega_{\mathrm{QED}}^{(\mathrm{plas})} = -\frac{e^3 T}{12\pi^4}\left[\int_0^\infty \frac{dp}{E_p}(2p^2 + m^2)n_p\right]^{3/2} + O(e^4) \tag{89}$$

It is interesting to take here the classical statistics limit. In this limit the distribution $n(p)$ is approximated by its Boltzmann counterpart

$$n_p \simeq e^{-(E_p - \mu)/T}$$

The integral in equation (85) can now be expressed by the Bessel function of the first order $K_1(z)$ and its derivative resulting in

$$\Omega_{\mathrm{QED}}^{(\mathrm{plas})} = -\frac{e^3 T}{12\pi^4} e^{3\mu/2T}\left[mTK_1\left(\frac{m}{T}\right) - m^2 K_1'\left(\frac{m}{T}\right)\right]^{3/2} \tag{90}$$

Using the large argument approximation for $K_1(z)$

$$K_1(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

we obtain the nonrelativistic value $T \ll m$, $\mu' \ll m$

$$\Omega_{\mathrm{QED}}^{(\mathrm{plas})} = -\frac{e^3 (m^9 T^7)^{1/4}}{12\pi^4}\left(\frac{\pi}{2}\right)^{3/4} e^{3\mu'/2T} \tag{91}$$

This is the Debye–Hückel formula obtained as early as 1923 by a purely classical treatment (Debye and Hückel, 1923).

4.4. Thermodynamic Quantities

Let us here summarize the results found above. The thermodynamic potential for arbitrary m_f and μ_f is obtained up to $O(g^4 \ln g)$ by summing the contributions from equations (61), (67), (79), and (83). However, we shall now concentrate on the ultrarelativistic limit, $m_f = 0$ (but keeping the chemical potentials arbitrary) in order to simplify the discussion. In this limit, the pressure is given by (the modified plasmon has not yet been taken into account in these equations)

$$\begin{aligned}
p = & \left[(N^2-1+7NN_f/4)T^4 + 15\sum_f (T^2\theta_f^2 + \theta_f^4) \right] \pi^2 \Big/ 45 \\
& - (N^2-1)\left[(N+5N_f/4)T^4 + 9\sum_f (T^2\theta_f^2+\theta_f^4) \right] g^2 \Big/ 144 \\
& + (N^2-1)\left[(2N+N_f)T^2/6 + \sum_f \theta_f^2 \right]^{3/2} g^3 T \Big/ 12\pi \\
& + N(N^2-1)\left[(2N+N_f)T^2/6 + \sum_f \theta_f^2 \right] T^2 g^4 \ln g \Big/ 32\pi^2 + O(g^4)
\end{aligned} \tag{92}$$

Here $\theta_f^2 = \mu_f^2/2\pi^2$. The entropy density $s = \partial p/\partial T$ and the specific heat $c_V = \partial s/\partial T$ can now be obtained, from equation (88) yielding

$$\begin{aligned}
s = & \frac{\pi^2}{45}\left\{ [4(N^2-1)+7NN_f]T^3 + 30T\sum_f \theta_f^2 \right\} \\
& - \frac{N^2-1}{144}\left[(4N+5N_f)T^3 + 18T\sum_f \theta_f^2 \right] g^2 \\
& + \frac{N^2-1}{36\pi}\left[2(2N+N_f)T^2 + 3\sum_f \theta_f^2 \right]\left[\frac{(2N+N_f)T^2}{6} + \sum_f \theta_f^2 \right]^{1/2} g^3 \\
& + \frac{N^2-1}{48\pi^2} N \left[(2N+N_f)T^3 + 3T\sum_f \theta_f^2 \right] g^4 \ln g + O(g^5)
\end{aligned} \tag{93}$$

and

$$\begin{aligned}
c_V = & \frac{\pi^2}{15}\left\{ [4(N^2-1)+7NN_f]T^2 + 10\sum_f \theta_f^2 \right\} \\
& - \frac{N^2-1}{48}\left[(4N+5N_f)T^2 + 6\sum_f \theta_f^2 \right] g^2
\end{aligned}$$

$$+\frac{N^2-1}{72\pi}\frac{(2N+N_f)[2(2N+N_f)T^2+5\,\Sigma_f\,\theta_f^2]}{[(2N+N_f)T^2/6+\Sigma_f\,\theta_f^2]^{1/2}}\,Tg^3$$

$$+\frac{N^2-1}{12\pi^2}\,N\left[(2N+N_f)T^2+\sum_f\theta_f^2\right]g^4\ln g+O(g^4) \tag{94}$$

The average number density $n_f = n_{q_f} - n_{\bar{q}_f}$ representing the excess of quarks or antiquarks is

$$n_f=\frac{\partial p}{\partial\mu_f}=\frac{2\pi^2}{3}\,N(\theta_fT^2+2\theta_f^3)-\frac{N^2-1}{8}\,(\theta_fT^2+2\theta_f^3)g^2$$

$$+\frac{N^2-1}{4\pi}\,T\left[(2N+N_f)T^2/6+\sum_f\theta_f^2\right]^{1/2}\theta_fg^3$$

$$+\frac{N^2-1}{16\pi^2}\,NT^2\theta_fg^4\ln g+O(g^4) \tag{95}$$

The reader should note that in equations (92)-(95) we have not yet taken into account the renormalization group improved coupling constant $g = g(T, \mu)$. That will be discussed in Section 6. However, the differentiation of the coupling constant with respect to the temperature or chemical potential in equations (92)-(95) will yield a contribution of the order $O(g^4)$ up to which order we have not extended our calculations.

5. GROUND STATE OF FERMION GAS

At zero temperature the fermion states in the Fermi sphere are filled up to the Fermi energy μ_i (the index i corresponds to the flavor i). This is seen by taking the zero-temperature limit of the Fermi–Dirac distribution

$$\begin{aligned} n_i^-(p) &\to \theta(\mu_i - E_p) \\ n_i^+(p) &\to 0 \end{aligned} \tag{96}$$

This ground state of the Fermi gas is characterized by the zero-temperature limit of the thermodynamic potential, $\Omega(T=0)$. The energy density of this ground state is given by

$$\varepsilon = \Omega(T=0) + \bar{\mu}\cdot\bar{N} \tag{97}$$

where $\bar{\mu}$ and $\bar{N}$ represent the chemical potential and average number density vectors, their components corresponding different flavors.

In this section we shall discuss this ground state of the quark system in QCD (and of the electron system in QED). Our discussion here is, for the overlapping parts, essentially the same as that of Freedman and McLerran (1977), and the reader is urged to consult this reference for more details.

In Section 5.1, the zero-temperature limit of the ideal gas formula and the exchange correction are obtained and those contributions are examined in different limits. In Section 5.2 we give a detailed calculation of the plasmon term (of the order $g^4 \ln g$) retaining the masses nonzero. Finally, in Section 5.3, we discuss the fourth-order correction.

5.1. Ideal Gas Approximation and Exchange Term at Zero Temperature

Using the formula (96) we can find the ideal gas approximation from equation (61)

$$\Omega_0 = -\frac{N}{12\pi^2}\sum_i \left[\mu_i(\mu_i^2 - m_i^2)^{1/2}\left(\mu_i^2 - \frac{5}{2}m_i^2\right) + \frac{3}{2}m_i^4 \ln \frac{\mu_i + (\mu_i^2 - m_i^2)^{1/2}}{m_i}\right] \tag{98}$$

The exchange term is obtained similarly from equation (67). Note that only the quark loops (Figures 3a and 3e) contribute at zero temperature. The result, after performing the integration, is

$$\Omega_{\text{exch}} = \frac{N^2-1}{64\pi^4} g^2 \sum_i \left\{3\left[\mu_i(\mu_i^2 - m_i^2)^{1/2} - m_i^2 \ln \frac{\mu_i + (\mu_i^2 - m_i^2)^{1/2}}{m_i}\right]^2 - 2(\mu_i^2 - m_i^2)^2\right\} \tag{99}$$

In the ultrarelativistic limit ($\mu_i \gg m_i$), the above formulas reduce to

$$\Omega_0 + \Omega_{\text{exch}} = -\left(\frac{N}{12\pi^2} - \frac{N^2-1}{64\pi^4} g^2\right)\sum_i \mu_i^4 \tag{100}$$

This would have been obtained also from equations (62) and (68) by putting $T = 0$. The nonrelativistic limit is given by

$$\Omega_0 + \Omega_{\text{exch}} = -\frac{N}{15\pi^2}\sum_i \frac{(\mu_i^2 - m_i^2)^{5/2}}{m_i} - \frac{N^2-1}{32\pi^4} g^2 \sum_i (\mu_i^2 - m_i^2)^2 \tag{101}$$

Note that Ω_{exch} is positive in the ultrarelativistic limit, but has a negative sign in the nonrelativistic limit. [This also occurs at $T \neq 0$; see equations (68), (69). The contribution corresponding to flavor i changes its sign at the value

$$\mu_i = 2.723 m_i$$

At a given m_i the most negative value of the exchange contribution (corresponding the flavor i) is

$$\Omega^i_{\text{exch}}(\mu_i = 2.188 m_i) = -4.511 m_i^4 \frac{N^2-1}{64\pi^4} g^2$$

5.2. Plasmon Effect

In this section we shall evaluate the ring diagrams of Figure 4. In these diagrams the matter part of the lowest-order polarization tensor insertion comprises only the quark loop (Figure 1a), at zero temperature. The vacuum part $\Pi^{(2)}_{\text{vac}}$ gives here, like at $T \neq 0$, corrections of the order $O(g^4)$. Hence, we need take into account only the μ-dependent part $\Delta\Pi$.

There are no differences between QED and QCD in the topology of the diagrams under consideration. However, in QCD, different polarization tensor insertions can have fermion loops of different flavors. Thus the result will be mixed in the flavors. Hence, we will first consider QED in order to see more clearly the structure of this plasmon term as a function of the ratio μ/m. Later we shall give the correspondent equation for several flavors.

The diagrams of Figure 4 can be summed giving

$$\Omega_{\text{pl}} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left\{\ln\left(1+\frac{\Delta\Pi_{00}}{\bar{q}^2}\right)+2\ln\left[1+\frac{1}{2}\left(\frac{\Delta\Pi^{\mu}_{\mu}}{q^2}-\frac{\Delta\Pi_{00}}{\bar{q}^2}\right)\right]-\frac{\Delta\Pi^{\mu}_{\mu}}{q^2}\right\} \tag{102}$$

Note that here, unlike at $T \neq 0$, we cannot expand the contribution of spatial photons up to $O(e^4)$, because the fourth order diagram consisting of spatial photons is infrared divergent here.

The evaluation of the integral (102) is more easily performed by using Euclidean spherical coordinates, equation (50). Using the shorthand

$$\begin{aligned}
\Lambda_1(q^2, \phi) &= (1/\sin^2\phi)\Delta\Pi_{00}(q^2, \phi) \\
\Lambda_2(q^2, \phi) &= \tfrac{1}{2}[\Delta\Pi^{\mu}_{\mu}(q^2, \phi) - (1/\sin^2\phi)\Delta\Pi_{00}(q^2, \phi)]
\end{aligned} \tag{103}$$

equation (102) becomes after the trivial integrations

$$\begin{aligned}
\Omega_{\text{pl}} = \frac{1}{(2\pi)^3}\int_0^{\infty} dq^2\, q^2 \int_0^{\pi/2} d\phi\, \sin\phi \, &\left\{\ln\left[1+\frac{\Lambda_1(q^2,\phi)}{q^2}\right]\right. \\
&\left. +2\ln\left[1+\frac{\Lambda_2(q^2,\phi)}{q^2}\right]-\frac{\Lambda_1(q^2,\phi)}{q^2}-\frac{2\Lambda_2(q^2,\phi)}{q^2}\right\}
\end{aligned} \tag{104}$$

In order to extract the dominant contribution we write equation (104) as

$$\begin{aligned}
\Omega_{\text{pl}} = \frac{1}{(2\pi)^3}\int_0^{\infty} dq^2\, q^2 \int_0^{\pi/2} d\phi\, \sin\phi \, &\left\{\ln\left[1+\frac{\Lambda_1(0,\phi)}{q^2}\right]\right. \\
&+2\ln\left[1+\frac{\Lambda_2(0,\phi)}{q^2}\right]-\frac{1}{2q^2(q^2+\mu^2)}[\Lambda_1^2(0,\phi)+2\Lambda_2^2(0,\phi)] \\
&\left. -\frac{1}{q^2}[\Lambda_1(0,\phi)+2\Lambda_2(0,\phi)]\right\}+O(e^4)
\end{aligned} \tag{105}$$

Here we have neglected infrared convergent contributions of the order $O(e^4)$. Note the similar arbitrariness in the third term of equation (105), as we found in equation (81). The term μ^2 in the denominator can be replaced by any constant (different from zero and independent of e). Here, like at $T \neq 0$, the different choices give the same result apart from a contribution of the order $O(e^4)$.

The integrations over q^2 can now be done exactly. Omitting again contributions of the order $O(e^4)$ we get

$$\Omega_{\mathrm{pl}} = \frac{\ln e^2}{(2\pi)^3} \int_0^{\pi/2} d\phi \sin\phi \left[\frac{1}{2}\Lambda_1^2(0, \phi) + \Lambda_2^2(0, \phi)\right] \tag{106}$$

Now we insert equations (57), (59) to equations (103) obtaining

$$\Lambda_1(0, \phi) = \frac{e^2}{\pi^2}\left\{\mu(\mu^2 - m^2)^{1/2} - \mu^2 \cot\phi \arctan\left[\frac{(\mu^2 - m^2)^{1/2}}{\mu \cot\phi}\right]\right\} \tag{107}$$

$$\Lambda_2(0, \phi) = \frac{e^2}{\pi^2}\left\{\mu(\mu^2 - m^2)^{1/2}\left(1 - \frac{1}{\sin^2\phi}\right) - \left(m^2 - \frac{\mu^2}{\sin^2\phi}\right)\cot\phi \arctan\left[\frac{(\mu^2 - m^2)^{1/2}}{\mu \cot\phi}\right]\right\} \tag{108}$$

Inserting equation (107), (108) to (106) we can write after some lengthy manipulation the result in the form

$$\Omega_{\mathrm{pl}} = \frac{e^4 \ln e^2}{128\pi^6} \mu^4 f_1\left(\frac{m}{\mu}\right)$$

where

$$\begin{aligned}\mu^4 f_1(m/\mu) = {} & (6 - 4\ln 2)\mu(\mu^2 - m^2)^{3/2} - 5\mu^2(\mu^2 - m^2) \\ & + 4\mu^3(\mu^2 - m^2)^{1/2} \ln\frac{\mu + (\mu^2 - m^2)^{1/2}}{\mu} \\ & + 6\mu m^2(\mu^2 - m^2)^{1/2} \ln\frac{\mu + (\mu^2 - m^2)^{1/2}}{2^{5/3}\mu} \\ & - (4m^2\mu^2 + m^4)\left(\ln\frac{\mu + (\mu^2 - m^2)^{1/2}}{m}\right)^2 \\ & + m^2\mu\frac{4\mu^2 + m^2}{(\mu^2 - m^2)^{1/2}} I(a).\end{aligned} \tag{109}$$

Here the function $I(a)$ is defined by

$$I(a) = \int_1^\infty \frac{dx}{a^2x^2 - 1} \ln\frac{x + 1}{x - 1} \tag{110}$$

where

$$a = \frac{\mu}{(\mu^2 - m^2)^{1/2}} \geq 1 \tag{111}$$

The function $I(a)$ cannot be expressed as a finite sum of elementary functions. However, we can get a power series in $1/a^2$ for it

$$I(a) = \sum_{n=1}^{\infty} \frac{1}{2n-1} [2 \ln 2 + \psi(n) - \psi(1)] \frac{1}{a^{2n}} \tag{112}$$

where $\psi(x)$ is the di-gamma function satisfying

$$\psi(n) - \psi(1) = \sum_{k=1}^{n-1} \frac{1}{k} \tag{113}$$

Note that in the nonrelativistic limit $a \gg 1$. Hence, our series (112) is a nonrelativistic expansion for $I(a)$. For the ultrarelativistic limit the last term in equation (109) containing the function $I(a)$ is small, however, because it is proportional to $m^2 \ln^2 m/\mu$. When we take these two limits of equation (109) we find them to be in agreement with the earlier results. Firstly the ultrarelativistic limit is (Akhiezer and Peletminsky, 1960; Freedman and McLerran, 1977)

$$\Omega_{\text{pl}}(m=0) = \frac{e^4 \ln e^2}{128\pi^6} \mu^4 \tag{114}$$

On the other hand the nonrelativistic limit is

$$\Omega_{\text{pl}} = \frac{e^4 \ln e^2}{48\pi^6} (1 - \ln 2)\mu(\mu^2 - m^2)^{3/2} \tag{115}$$

In this limit only the first term of equation (106) contributes. This part is associated with the timelike photons and its nonrelativistic limit corresponds to the interaction of the electrons through the instantaneous Coulomb field. The second part corresponds to the interaction of the electrons with the magnetic field induced by the other moving electrons. Its contribution is reduced in the nonrelativistic limit by a factor of μ'/m.

$f_1(m/\mu)$ is shown as a function of μ/m in Figure 5. When $\mu \gtrsim 3m_e$, the function differs from the ultrarelativistic limit by less than 18%.

In QED, these higher-order corrections are less important since the coupling constant is small. The situation is very different in QCD, where the coupling constant is of the order unity. What we need to do in QCD is to replace $e^2 \to g^2/2$ and sum over the flavors in the definition of the functions Λ_i [equations (107), (108)]. This summation causes some extra complexities in the ϕ integration of equation (106). The calculations are lengthy and tedious, and we shall omit them here. After some manipulation we get the

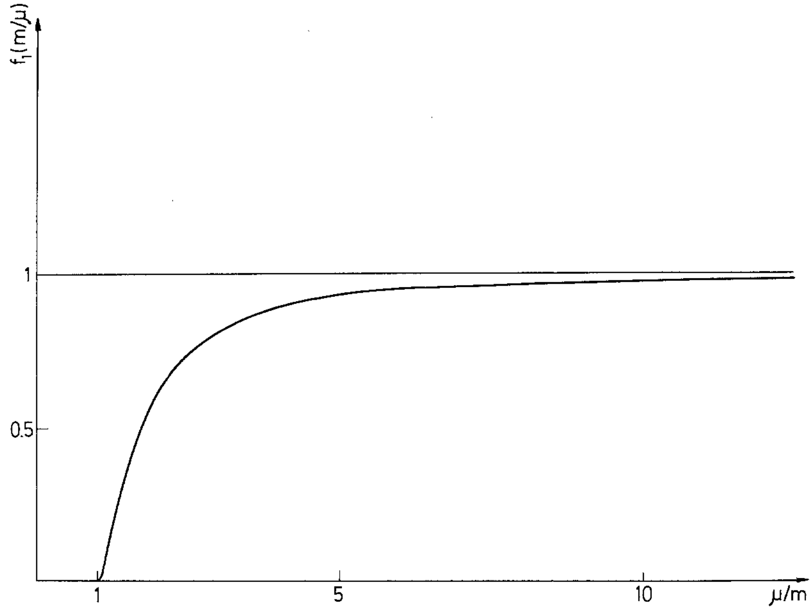

Fig. 5. The function f_1 as a function of μ/m.

plasmon term in the form

$$\Omega_{\mathrm{pl}}^{\mathrm{QCD}}=\frac{g^4\ln g^2}{256\pi^6}\sum_{i,j}\frac{\mu_i^2\mu_j^2}{a_ia_j}\left\{-\frac{5}{2}-\frac{(a_i^2-1)(a_j^2-1)}{8a_ia_j}\ln\frac{a_i+1}{a_i-1}\ln\frac{a_j+1}{a_j-1}\right.$$
$$+\left[\frac{3}{2a_i}-\frac{a_i^2-1}{4a_i}a_j\ln\frac{a_i+1}{a_i-1}\ln\frac{a_j+1}{a_j-1}+\frac{5a_i^2-3}{2a_i}\ln\frac{a_i+1}{a_i+a_j}\right.$$
$$+\left(\frac{a_j^2-1}{2a_j}a_i+\frac{a_i^2-1}{2a_i}a_j+\frac{(a_i^2-1)(a_j^2-1)}{4a_ia_j}\right)$$
$$\left.\left.\times\int_{a_j}^{\infty}\frac{dx}{x^2-1}\ln\left|\frac{a_i+x}{a_i-x}\right|\right]+[i\leftrightarrow j]\right\} \tag{116}$$

where

$$a_i=\frac{\mu_i}{(\mu_i^2-m_i^2)^{1/2}}$$

It can be easily checked that the above formula leads in the one-flavor case to equation (109), as it should. Furthermore, we find that the zero-mass limit is in agreement with the earlier results (Freedman and McLerran, 1977); putting $a_i=1$ we get

$$\Omega_{\mathrm{pl}}(m_i=0)=\frac{g^4\ln g^2}{512\pi^6}(\bar{\mu}^2)^2 \tag{117}$$

We can also derive from equation (116) a nonrelativistic expansion as we did in QED case. However, it is not worth doing this because the situation where all the flavors are nonrelativistic does not correspond to any known physical circumstances. A more realistic case is obtained, if we consider two massless flavors (u and d quarks) and take the mass of the strange quark into account, and neglect the heavier flavors. Equation (117) then reads

$$\Omega_{\mathrm{pl}} = \frac{g^4 \ln g^2}{512\pi^6}\left[(\mu_u^2+\mu_d^2)^2 + \mu_s^2 f_1\left(\frac{m_s}{\mu_s}\right) + 2(\mu_u^2+\mu_d^2)\mu_s^2 f_2\left(\frac{m_s}{\mu_s}\right)\right] \quad (118)$$

Here $f_1(m/\mu)$ is given by equation (109) and $f_2(m/\mu)$ is defined by

$$f_2\left(\frac{m_s}{\mu_s}\right) = \frac{2}{a}\left\{\frac{3}{2a} - 1 + \ln\frac{2}{a+1} + \frac{a^2-1}{4a}\left[\ln\frac{a+1}{a-1}\ln\frac{a^2-1}{4} + 2aI(a)\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(\ln\frac{a+1}{a-1}\right)^2 + \int_1^a \frac{dx}{x^2-1}\left(\ln\frac{a+x}{a-x} - x\ln\frac{a+1}{a-1}\right)\right]\right\} \quad (119)$$

where $I(a)$ is given by (110) and

$$a = \mu_s/(\mu_s^2 - m_s^2)^{1/2}$$

The function $f_2(m/\mu)$ is drawn in Figure 6. If we compare the functions $f_1(m/\mu)$ and $f_2(m/\mu)$, we note that f_2 reaches its ultrarelativistic value more

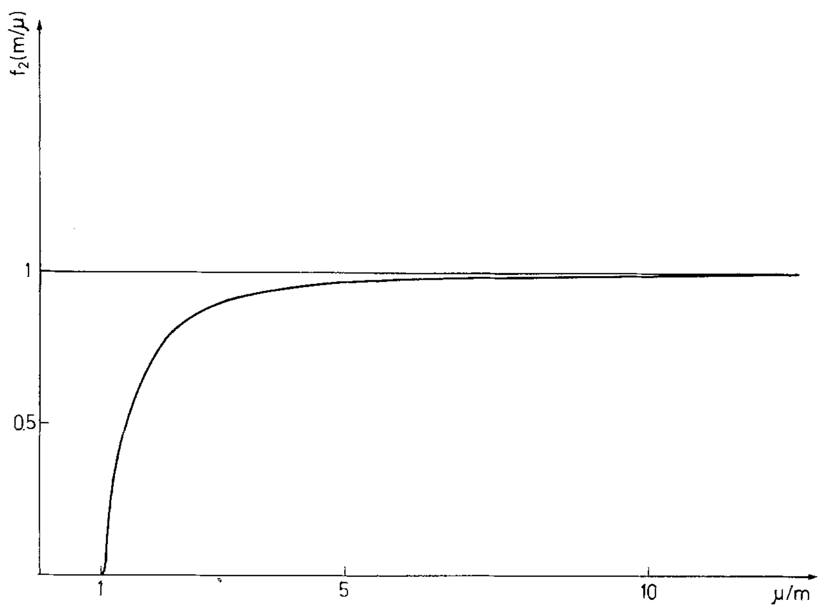

Fig. 6. The function f_2 as a function of μ/m.

rapidly than f_1. When $\mu_s \gg m_s$ we get from equation (119)

$$\Omega_{\rm pl} = \frac{g^4 \ln g^2}{512\pi^6}\left[(\bar{\mu}^2)^2 - \left(5 - \frac{\pi^2}{3}\right)\bar{\mu}^2 m_s^2 + O(m_s^4)\right] \tag{120}$$

This equation, together with equations (98), (99), represent the situation in the plasma phase in the neutron star core. There $\mu_s = \mu_d \simeq \mu_u \simeq 500$ MeV. Hence, if we use $m_s \simeq 150$ MeV, our expansion parameter in equation (120) is

$$m_s^2/\mu_s^2 \simeq 0.09$$

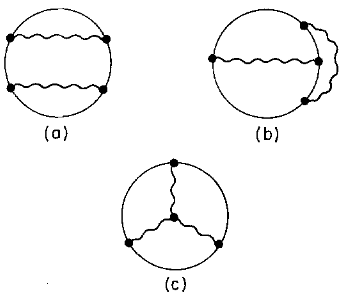

Fig. 7. The $O(g^4)$ diagrams not included in the ring diagrams. In QED there are only the diagrams (a, b). The counterterm diagrams are not shown explicitly.

which is rather small, and we see from Figures 5 and 6 that the curves comply with equation (120) rather well at large μ_s.

5.3. Survey of the Higher-Order Calculations

In the order $O(g^4)$, QED and QCD differ essentially because the non-Abelian structure of QCD appears for the first time in this order, and hence the topological structure of the diagrams is no longer the same for these two theories (see Figure 7). Also the existence of more than one flavor gives a rather complicated structure to this fourth-order term in QCD.

The evaluation of the fourth-order term also includes, besides the diagrams of Figure 7, all the $O(g^4)$ terms of the ring diagrams (Figure 4), which we ignored in the previous subsection. At the three-loop level [order $O(g^4)$] there exists only calculations in the massless limit, and (for QED) in the nonrelativistic limit.

Combining all contributions, the QED thermodynamic potential is at $m = 0$ (Freedman and McLerran, 1977)

$$\Omega = -\frac{\mu^4}{12\pi^2}\left[1 - \frac{3}{2}\frac{\alpha}{\pi} - \frac{3}{2}\left(\frac{\alpha}{\pi}\right)^2 \ln\frac{\alpha}{\pi} - \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \ln\frac{\mu^2}{\mu_0^2} + (2.11 \pm 0.12)\left(\frac{\alpha}{\pi}\right)^2\right] \tag{121}$$

Here α is the fine structure constant $\alpha = e^2/4\pi$ and μ_0 is the Euclidean subtraction point.

For QCD the corresponding formula is (Freedman and McLerran, 1977)

$$\Omega = -\frac{1}{4\pi^2}\left(\sum_i \mu_i^4\left\{\frac{N}{3} - (N^2-1)\frac{\alpha_c}{4\pi} + (N^2-1)\frac{11N-2N_f}{3}\left(\frac{\alpha_c}{4\pi}\right)^2 \ln\frac{\mu_i^2}{\mu_0^2}\right.\right.$$
$$\left.\left.+ (N^2-1)\left(\frac{\alpha_c}{4\pi}\right)^2\left[-2.250N + 0.409N_f - 3.697 - (4.24\pm 0.12)\frac{1}{N}\right]\right\}\right.$$
$$\left. - (\bar{\mu}^2)^2(N^2-1)\left(\frac{\alpha_c}{4\pi}\right)^2\left(2\ln\frac{\alpha_c}{4\pi} - 0.476\right) - \left(\frac{\alpha_c}{4\pi}\right)^2 (N^2-1)F(\mu)\right) \quad (122)$$

Here $\alpha_c = g^2/4\pi$ and the function $F(\mu)$ is given by

$$F(\mu) = -2\sum_i \bar{\mu}^2\mu_i^2 \ln\frac{\mu_i^2}{\bar{\mu}^2} + \sum_{i>j}\left[\frac{2}{3}(\mu_i-\mu_j)^4 \ln\frac{|\mu_i^2-\mu_j^2|}{\mu_i\mu_j}\right.$$
$$\left. + \frac{8}{3}\mu_i\mu_j(\mu_i^2+\mu_j^2)\ln\frac{\mu_i^2+\mu_j^2}{\mu_i\mu_j} - \frac{2}{3}(\mu_i^4-\mu_j^4)\ln\frac{\mu_i}{\mu_j}\right] \quad (123)$$

It is worth noting that the contribution of the order $O(g^4)$ depends on the renormalization scheme. (Equivalently: it depends on the definition of the charge.) Note also that a different choice of gauge leads to a different definition of the coupling constant and hence the result depends on the gauge as a part of the renormalization prescription. The above formulas have been calculated in the Landau gauge using the momentum space substraction scheme. (For the relation between the couplings in different schemes, see Celmaster and Sivers, 1981.)

In QED it is interesting to consider the nonrelativistic limit. There the energy per particle, which is related to the thermodynamic potential by the equation

$$E/N = \mu - \Omega \bigg/ \frac{\partial\Omega}{\partial\mu} \quad (124)$$

has been calculated up to the order $O(e^6 \ln e^2)$. Reexpressing equation (124) as a function of N/V instead of μ and introducing the dimensionless variable

$$r_s = \left(\frac{4\pi}{3}\frac{N}{V}\right)^{1/3} m\alpha \quad (125)$$

we can write equation (123) as

$$\frac{E'}{N} = \frac{a}{r_s^2} + \frac{b}{r_s} + c\ln r_s + d + er_s\ln r_s + \cdots \quad (126)$$

where $E'/N = E/N - m$. The three first coefficients can be obtained by substituting equations (101) and (115) into equation (124) and using equations (5) and (125). The fourth and fifth terms have been calculated numerically (Gell-Mann and Bruckner, 1957; Can and Maradudin, 1964) (see also Onsager, Mittag, and Stephen, 1966). The fifth term in (126) ~ $r_s \ln r_s$ comes from the summation of the next to leading (infrared) divergences in the ring diagrams of Figure 4. In expressing the energy per particle by the binding energy of the hydrogen atom ($E_0 = 1$ Rydberg $\simeq 13.6$ GeV), we get (Fetter and Walecka, 1971)

$$\frac{E'}{N} = E_0 \left[\frac{3}{5} \left(\frac{9\pi}{4} \right)^{2/3} \frac{1}{r_s^2} - \frac{3}{2\pi} \left(\frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s} + \frac{2}{\pi^2} (1 - \ln 2) \ln r_s \right.$$
$$\left. - 0.094 + 0.018 r_s \ln r_s + O(r_s) \right] \tag{127}$$

It is worth noting that in this nonrelativistic limit (also true in QCD) the expansion parameter is

$$r_s \sim \frac{me^2}{(\mu^2 - m^2)^{1/2}}$$

which is small only if the chemical potential is not too close to the mass; in other words, if the system is not too nonrelativistic. In the electron gas of a metal this does not cause any serious problem because the density of the electron gas (and consequently μ) is fixed by the properties of the positively charged background lattice, and hence the difference $\mu - m$ cannot ever vanish.

However, if we consider QCD, we have several flavors and it is possible that one of the chemical potentials in some system (for example, in neutron star) can pass the threshold $\mu_f \to m_f$. If we extrapolate this behavior of the nonrelativistic limit to higher orders, we find that in the order g^{12} the contribution will be

$$\sim \frac{g^{12} m^5}{(\mu^2 - m^2)^{1/2}}$$

Hence, all the terms of order g^{12} or higher diverge when μ goes to m. This is probably a reflection of a nonanalytic behavior (in g^2) of the energy and the other thermodynamic quantities near the threshold.

6. EQUATION OF STATE AND PHASE TRANSITION

In the preceding chapters we calculated perturbatively the thermodynamic potential. In order to improve our calculations we must take into account the renormalization group improved coupling constant $\alpha = \alpha(T/\Lambda_{\rm QCD}, \mu/\Lambda_{\rm QCD})$. For simplicity, we have neglected the chemical potential, and thus have for massless QCD (see Caswell, 1974)

$$\begin{aligned}\frac{N\alpha(T)}{\pi} &\equiv \frac{Ng^2(T)}{4\pi^2}\\ &= \frac{6}{(11N-2N_f)\ln T/\Lambda}\\ &\quad - g\frac{(34N^2-10NN_f-3N_f(N^2-1)/N)\ln\ln T/\Lambda}{(11N-2N_f)^3\ln^2 T/\Lambda}\\ &\quad + O\left(\frac{1}{\ln^2 T/\Lambda}\right)\end{aligned} \tag{128}$$

As we found in Section 4, the perturbative expansion for the thermodynamic potential (equally for the pressure $p=-\Omega$) is not analytic in the coupling constant α, but has nonanalytic terms of type $\sim\alpha^{n+1/2}$ and also logarithmic-type contributions $\sim\alpha^{n/2}\ln\alpha$. Hence we have in general the expansion for the pressure

$$p/T^4 = c + \sum_{n=0}^{\infty} a_n[\alpha(T)/\pi]^{1+n/2} + \sum_{b=0}^{\infty} b_n[\alpha(T)/\pi]^{2+n/2}\ln\alpha(T)/\pi \tag{129}$$

Note that in QED the logarithmic terms are absent.

For QCD the coefficients c, a_0, a_1, and b_0 are known. If the quarks are massive, the coefficients are functions of the ratios m_f/T. To be more specific, we put $N=3$ and regard u, d, and s quarks as massless and ignore the contributions of the heavier flavors. After substituting the coefficient calculated in Section 4, we get

$$\begin{aligned}p = \pi^2T^4\bigg\{&\frac{19}{36} - \frac{3\alpha(T)}{2\pi} - \left(\frac{8}{3\sqrt{2}}\right)\left[\frac{3\alpha(T)}{\pi}\right]^{3/2}\\ &+\frac{2}{3}\left[\frac{3\alpha(T)}{\pi}\right]^2\ln\frac{\alpha(T)}{\pi} + O(\alpha^2(T))\bigg\}\end{aligned} \tag{130}$$

The equation of state can be obtained by using the equation

$$\varepsilon = T^2\frac{\partial}{\partial T}\left(\frac{p}{T}\right) \tag{131}$$

yielding

$$\varepsilon = 3p + \pi^2T^4\left[\frac{3}{4}\left(\frac{3\alpha}{\pi}\right)^2 - 2\sqrt{2}\left(\frac{3\alpha}{\pi}\right)^{5/2} - 2\left(\frac{3\alpha}{\pi}\right)^3\ln\frac{\alpha}{\pi}\right] + O(\alpha^3) \tag{132}$$

The sound velocity v_s is obtained by

$$v_s^2=\frac{dp}{d\varepsilon}=\frac{1}{3}-\frac{9}{19}\left(\frac{3\alpha}{\pi}\right)^2+\frac{24\sqrt{2}}{19}\left(\frac{3\alpha}{\pi}\right)^{5/2}+\frac{24}{19}\left(\frac{3\alpha}{\pi}\right)^3\ln\frac{\alpha}{\pi}+O(\alpha^3) \quad (133)$$

We now eliminate the temperature from equations (130) and (132) in order to get the equation of state in the form $p=p(\varepsilon)$. The result is

$$p=\frac{\varepsilon}{3}\left[1-\frac{10.1}{\ln^2\varepsilon/\varepsilon_0}+\frac{46.7}{\ln^{5/2}\varepsilon/\varepsilon_0}-\frac{10.0\ln\ln\varepsilon/\varepsilon_0}{\ln^3\varepsilon/\varepsilon_0}+O\left(\frac{1}{\ln^3\varepsilon/\varepsilon_0}\right)\right] \quad (134)$$

where $\varepsilon_0\sim\Lambda^4$.

Equations (130)-(134) are evaluated consistently up to the order the perturbative calculations in Section 4 allow. However, as we discussed in Section 4, these results are reliable only when g^2 is extremely small, i.e., when T is extremely high compared to Λ. When the temperature is lowered, the plasmon terms overwhelm the exchange correction and finally even the ideal gas term. Hence at these lower temperatures we must use the "modified plasmon," which takes into account the dominant contribution beyond the order $O(\alpha^2\ln\alpha)$. Equation (130) then reads

$$\begin{aligned} p=\pi T^4\Bigg(&\frac{19}{36}-\frac{3\alpha(T)}{2\pi}+\frac{8}{3\sqrt{2}}\left\{\frac{3\alpha(T)}{\pi}\left[1-\frac{5}{4\sqrt{2}}\left(\frac{3\alpha(T)}{\pi}\right)^{1/2}\right]\right\}^{3/2}\\ &+\frac{2}{3}\left(\frac{3\alpha(T)}{\pi}\right)^2\left[1-\frac{5}{4\sqrt{2}}\left(\frac{3\alpha(T)}{\pi}\right)^{1/2}\right]\\ &\times\ln\left\{\frac{\alpha(T)}{\pi}\left[1-\frac{5}{4\sqrt{2}}\left(\frac{3\alpha(T)}{\pi}\right)^{1/2}\right]\right\}\Bigg) \end{aligned} \quad (135)$$

The pressure curves corresponding to different orders of the perturbation theory have been drawn in Figure 8. Using these curves, we shall now try to estimate the critical temperature where the phase transition takes place. However, strictly speaking, we are not able to extract any accurate information about the phase transition without knowing the equation of state of the hadronic phase. Nevertheless, what we shall do is to try to get at least a lower bound for the transition temperature.

The point where the curve (p_0+p_α) crosses zero has been used for estimating the phase transition temperature (Kalashnikov, 1984). This yields

$$T_c \gtrsim 2.1\Lambda_{\text{QCD}} \quad (136)$$

The inequality is given here because the pressure in the hadronic phase is, of course, nonzero and hence the phase transition takes place before the quark phase pressure reaches zero. If we take into account all the calculated corrections up to $O(g^4\ln g)$ and use the modified plasmon we get similarly

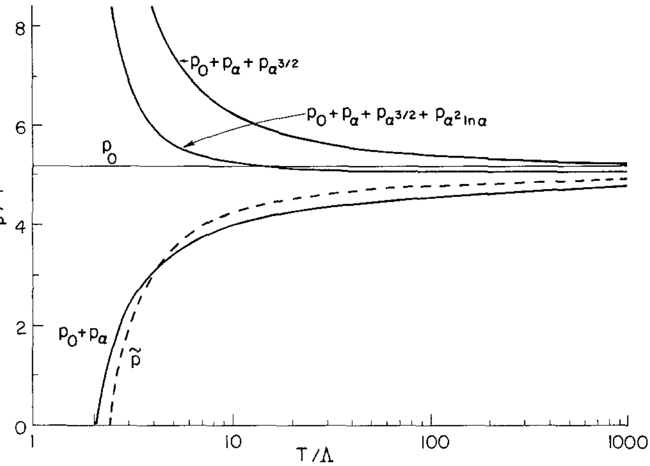

Fig. 8. The pressure as a function of the temperature. The subscripts denote the order of perturbation theory. $\tilde{p}$ is the pressure when the modified plasmon is used according to equation (135).

from the point where $\tilde{p}$ crosses zero

$$T_c \gtrsim 2.4\Lambda_{\text{QCD}} \tag{137}$$

It should be noted that the electric screening mass provides a similar estimate. Because $m_{\text{el}}=0$ in the confinement phase, we can estimate the phase transition temperature by ignoring the higher terms and setting the r.h.s. of equation (4.27) equal to zero. This gives a slightly smaller value for the transition temperature

$$T_c \approx 1.9\Lambda_{\text{QCD}} \tag{138}$$

However, all these equations (135)-(138) should be considered only as order of magnitude estimates. In any case, the curves (p_0+p_α) and $\tilde{p}$ support the idea that a phase transition from the quark phase to the hadronic phase occurs when the temperature is lowered. However, in order to get better information about the phase transition some information about the hadronic matter is also needed (see for example Kämpfer and Schultz, 1984; Suhonen, Dixit, and Turunen, 1984; and references quoted therein).

Note that we have here neglected the bag constant, because its inclusion will not change the qualitative picture.

7. SUMMARY AND OUTLOOK

In this paper we reviewed the finite temperature and density QCD and QED from the perturbative viewpoint. Using the formalism of the finite T

and μ field theory, we calculated the thermodynamic potential. We extended the calculations up to the order $O(g^4 \ln g)$, keeping the masses of the fermions arbitrary. This was done for $T \neq 0$, $\mu \neq 0$ in Section 4, and for $T = 0$, $\mu \neq 0$ in Section 5. The arbitrariness of the masses in this context is rather important because in the quark phase, which can possibly be created by means of heavy ion collisions, the temperature will be a few times the strange quark mass, and thus the strange quark cannot be regarded as either ultra- or nonrelativistic. The same is true in the cores of the neutron stars, where $T \simeq 0$, $\mu \gtrsim m_s$.

We further examined the phase transition and found that the perturbative calculations for QCD support the idea of two phases, unconfined and confined, separated from each other by a phase transition near $T_c \simeq 2\Lambda_{\mathrm{QCD}}$. However, the predictions that can be made when one uses only perturbative QCD are rather limited.

Concerning further studies one can ask what can be done next in the perturbation context. At $T = 0$, the thermodynamic potential is known in the massless limit up to the order $O(g^4)$. The next step will be the extension of the calculations up to $O(g^6 \ln g)$. For this extension, the evaluation of the polarization tensor is needed partly up to the two-loop level. Furthermore, the calculation with arbitrary masses should be extended up to $O(g^4)$. At $T \neq 0$ the order $O(g^4)$ is still lacking. This calculation will be more difficult than the corresponding calculation at zero temperature for two reasons. One is that the Fermi-Dirac and Bose-Einstein distributions make the integrals more complicated. The other is that unlike at $T = 0$, where all the ring diagrams that have infrared divergences are free of ultraviolet divergences, at $T \neq 0$ there exists a ring diagram having both singularities, which will cause some extra complexities.

Furthermore, if one goes to higher orders at $T \neq 0$, one must finally face the infrared problems associated with the "magnetic sector" of QCD (Linde, 1979, 1980; Gross, Pisarski, and Yaffe, 1981). If we consider the spatial part of the polarization tensor, we find that in $\Pi_{ij}(0, \bar{k})$, the first term at small $\bar{k}$, the term of order $\sim \bar{k}T$, has a negative sign [see equation (46)]. This "wrong" sign causes a tachyonic-like pole in the full propagator at nonzero $\bar{k}$, which must be cured before the order $O(g^6 \ln g)$ in the thermodynamic potential can be calculated. The reason is that the part of the $O(g^6 \ln g)$ contribution that arises from the ring diagrams having spatial gluons, equations (73), is not real unless the spurious pole is removed in some way (for example, by the higher loop contributions in the polarization tensor). The next order $O(g^6)$ is even more difficult. To calculate this one needs to know the magnetic mass that is still lacking (see the discussion by Toimela, 1985).

One interesting prospect for studying the field theory at finite T and μ will be the examination of the running coupling constant as a function of both variables T and μ. Some knowledge about the running coupling constant, when the ratio T/μ varies will be especially important in the context of heavy ion collisions, because in the central region $T \gg \mu$, whereas in the fragmentation regions $T \sim \mu$.

In this paper we have omitted several interesting topics at T, $\mu \neq 0$, that can be discussed in the context of the perturbation theory. One is the collective excitation spectrum of a QED or QCD plasma that can be obtained by setting the inverse of the full propagator equal to zero. The dispersion relations and damping constants can be calculated perturbatively (Fradkin, 1965; Kalashnikov, 1984; Kajantie and Kapusta, 1985; Klimov, 1981). Also not discussed here are the processes, where the initial particles are in thermal equilibrium but some of the final particles interact so weakly that they escape the finite size system and hence one can consider them to scatter into the vacuum. This is what happens in the QCD plasma formed in heavy ion collisions, when one examines the outcoming, electromagnetically interacting particles, the leptons, and photons (Feinberg, 1976; Shuryak, 1978; Domokos and Goldman, 1981; Kajantie and Miettinen, 1981, 1982; McLerran and Toimela, 1985).

In this paper we have used the imaginary time formalism. The reader should note that there exists also another possible formulation of perturbation theory at finite temperature, the real time formalism (Keldysh, 1964; Niemi and Semenoff, 1984; see also Umezawa, Matsumoto, and Tachiki, 1982; Semenoff and Umezawa, 1983). This method has the advantage that there is no frequency summation; in some sense one can say that the frequency sums there are done intrinsically, because the propagators include the Fermi-Dirac or Bose-Einstein distributions. Recall that in the imaginary time formalism the thermal distributions arise when the frequency sums are done. However, the price for getting rid of the frequency sums is that in the real time formalism the propagators and self-energies are 2×2 matrices and, moreover, the multiple δ functions here have to be regularized by replacing them by appropriate limiting expressions, which involve at least the same amount of work in the calculations as the frequency sums do.

The most important reason why we here have preferred the imaginary time formulation is that we have discussed only static quantities and thus we did not need to continue analytically our equations from imaginary to real energy. Hence, the advantage of the real time formalism having originally real energy is of no use here. Moreover, the nonanalytic terms (in g^2) discussed in Section 4.3 arose only from one term in the appropriate frequency sum, hence making the use of the imaginary time formulation very convenient. Nevertheless it would be interesting to see how the result obtained in Section 4.3, would arise in the real time formalism.

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APPENDIX

We derive here the results presented in equations (53), (54). Using equations (51), (52) we write $\Delta\Pi_{00}$ and $\Delta\Pi^{\mu}_{\mu}$ in the form (we delete here the sum over flavors)

$$\Delta\Pi^{\mu}_{\mu}=\frac{g^2}{\pi^2}\left[I_0+\frac{2m^2-q^2}{8q\sin\phi}I_1\right] \tag{A1}$$

$$\Delta\Pi_{00}=\frac{g^2}{2\pi^2}\left[I_0+\frac{1}{2q\sin\phi}\left(I_2-\frac{q^2}{4}I_1\right)-K\cot\phi\right] \tag{A2}$$

where

$$I_0=\int_0^{(\mu^2-m^2)^{1/2}}\frac{dp}{E_p}p^2 \tag{A3}$$

$$I_1=\int_0^{(\mu^2-m^2)^{1/2}}\frac{dp}{E_p}p\ln\frac{(2p\sin\phi+q)^2+4E_p^2\cos^2\phi}{(2p\sin\phi-q)^2+4E_p^2\cos^2\phi} \tag{A4}$$

$$I_2=\int_0^{(\mu^2-m^2)^{1/2}}dp\,pE_p\ln\frac{(2p\sin\phi+q)^2+4E_p^2\cos^2\phi}{(2p\sin\phi-q)^2+4E_p^2\cos^2\phi} \tag{A5}$$

$$K=\int_0^{(\mu^2-m^2)^{1/2}}dp\,p\arctan\left(\frac{pE_p\sin 2\phi}{E_p^2\cos 2\phi+m^2\sin^2\phi+q^{2/4}}\right) \tag{A6}$$

The first integral I_0 is trivial:

$$I_0=\frac{\mu(\mu^2-m^2)^{1/2}}{2}-\frac{m^2}{2}\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m} \tag{A7}$$

In the second integral I_1, we integrate by parts and use the substitution

$$x=\frac{p}{m+(p^2+m^2)^{1/2}}$$

This gives

$$I_1=\mu\ln\frac{4\mu^2\cos^2\phi+[2(\mu^2-m^2)^{1/2}\sin\phi+q]^2}{4\mu^2\cos^2\phi+[2(\mu^2-m^2)^{1/2}\sin\phi-q]^2}+J_1$$

$$J_1 = -8m \int_0^{[(\mu-m)/(\mu+m)]^{1/2}} dx \left(\frac{1+x^2}{1-x^2}\right)^2$$
$$\times\left[\frac{4x+\hat{q}(1-x^2)\sin\phi}{(4x\sin\phi+\hat{q}(1-x^2))^2+4(1+x^2)^2\cos^2\phi}-(\phi\to-\phi)\right], \qquad \hat{q}=q/m \tag{A8}$$

Factorizing as a product of two second-order polynomials the fourth-order polynomial in the denominator in the following way:

$$\begin{aligned}&[4x\sin\phi+\hat{q}(1-x^2)]^2+4(1+x^2)^2\cos^2\phi\\ &\quad=[\{[(4+\hat{q}^2)^{1/2}-2\sin\phi]x-\hat{q}\}^2+4\cos^2\phi]\\ &\qquad\times(\{[(4+\hat{q}^2)^{1/2}+2\sin\phi]x+\hat{q}\}^2+4\cos^2\phi)/(\hat{q}^2+4\cos^2\phi)\end{aligned} \tag{A9}$$

the integral becomes elementary and leads

$$\begin{aligned}J_1 &= q\sin\phi\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m}+\frac{(4m^2+q^2)^{1/2}\sin\phi}{2}\\ &\quad\times\ln\frac{\{[(4+\hat{q}^2)^{1/2}-2\sin\phi](\mu^2-m^2)^{1/2}-\hat{q}(\mu+m)\}^2+4(\mu+m)^2\cos^2\phi}{([(4+\hat{q}^2)^{1/2}+2\sin\phi](\mu^2-m^2)^{1/2}+\hat{q}(\mu+m))^2+4(\mu+m)\cos^2\phi}\\ &\quad+q\cos\phi\left\{\arctan\frac{[(4+\hat{q}^2)^{1/2}-2\sin\phi](\mu^2-m^2)^{1/2}-\hat{q}(\mu+m)}{2(\mu+m)\cos\phi}\right.\\ &\quad\left.-\arctan\frac{[(4+\hat{q}^2)^{1/2}+2\sin\phi](\mu^2-m^2)^{1/2}+\hat{q}(\mu+m)}{2(\mu+m)\cos\phi}\right\}\\ &\quad-(\phi\to-\phi)\end{aligned} \tag{A10}$$

Grouping together the logarithms and arcus functions we get

$$\begin{aligned}I_1 &= \mu\ln\frac{4\mu^2\cos^2\phi+[2(\mu^2-m^2)^{1/2}\sin\phi+q]^2}{4\mu^2\cos^2\phi+[2(\mu^2-m^2)^{1/2}\sin\phi-q]^2}\\ &\quad+2q\sin\phi\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m}+\frac{(q^2+4m^2)^{1/2}\sin\phi}{2}\\ &\quad\times\ln\frac{2\mu^2(2m^2+q^2)-2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}{2\mu^2(2m^2+q^2)+2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}\\ &\quad-q\cos\phi\arctan\left(\frac{\mu(\mu^2-m^2)^{1/2}\sin 2\phi}{\mu^2\cos 2\phi}\right)\end{aligned} \tag{A11}$$

The integral I_2 can be similarly modified to

$$I_2 = \frac{\mu^3}{3} \ln \frac{4\mu^2 \cos^2\phi + [2(\mu^2 - m^2)^{1/2} \sin\phi + q]^2}{4\mu^2 \cos^2\phi + [2(\mu^2 - m^2)^{1/2} \sin\phi - q]^2}$$

$$- \frac{8m^3}{3} \int_0^{[(\mu-m)/(\mu+m)]^{1/2}} dx \left(\frac{1+x^2}{1-x^2}\right)^4$$

$$\times \left[\frac{4x + \hat{q}(1-x^2)\sin\phi}{[4x\sin\phi + \hat{q}(1-x^2)]^2 + 4(1+x^2)^2\cos^2\phi} - (\phi \to -\phi) \right] \qquad \text{(A12)}$$

Using again equation (A9) we get finally

$$I_2 = \frac{1}{3} \left\{ \mu^3 \ln \frac{4\mu^2 \cos^2\phi + [2(\mu^2 - m^2)^{1/2} \sin\phi + q]^2}{4\mu^2 \cos^2\phi + [2(\mu^2 - m^2)^{1/2} \sin\phi - q]^2} + q\mu(\mu^2 - m^2)^{1/2} \sin\phi \right.$$

$$- q \frac{(1 + 2\cos 2\phi) q^2 + 6m^2 \cos 2\phi}{2} \sin\phi \ln \frac{\mu + (\mu^2 - m^2)^{1/2}}{m}$$

$$- \frac{(q^2 + 4m^2)^{1/2}[(1 + 2\cos 2\phi) q^2 - 4m^2 \sin^2\phi] \sin\phi}{8}$$

$$\times \ln \frac{2\mu^2(2m^2 + q^2) - 2\mu q(q^2 + 4m^2)^{1/2}(\mu^2 - m^2)^{1/2} - m^2(q^2 + 4m^2 \sin^2\phi)}{2\mu^2(2m^2 + q^2) + 2\mu q(q^2 + 4m^2)^{1/2}(\mu^2 - m^2)^{1/2} - m^2(q^2 + 4m^2 \sin^2\phi)}$$

$$+ \frac{q[(1 - 4\sin^2\phi) q^2 - 12m^2 \sin^2\phi] \cos\phi}{4}$$

$$\left. \times \arctan \left[\frac{\mu(\mu^2 - m^2)^{1/2} \sin 2\phi}{\mu^2 \cos 2\phi + m^2 \sin^2\phi + q^{2/4}} \right] \right\} \qquad \text{(A13)}$$

In the last integral K we integrate again by parts and now use the substitution

$$x = \frac{p}{(p^2 + m^2)^{1/2}}$$

This leads to

$$K = \frac{1}{2}(\mu^2 - m^2)\arctan\left[\frac{\mu(\mu^2-m^2)^{1/2}\sin 2\phi}{\mu^2\cos 2\phi + m^2\sin^2\phi + q^{2/4}}\right]$$
$$-2m^2\sin 2\phi\int_0^{(\mu^2-m^2)^{1/2}/\mu}$$
$$\times dx\frac{x^2}{1-x^2}\frac{(1+x^2)(\hat{q}^2+4\cos^2\phi)-4x^2\cos 2\phi}{[(1-x^2)(\hat{q}^2+4\cos^2\phi)+4x^2\cos 2\phi]^2+16x^2\sin^2 2\phi} \quad \text{(A14)}$$

After the rearrangement in the denominator

$$[(1-x^2)(\hat{q}^2+4\cos^2\phi)+4x^2\cos 2\phi]^2+16x^2\sin^2 2\phi$$
$$=\{[(\hat{q}^2+4\sin^2\phi)x-\hat{q}(\hat{q}^2+4)^{1/2}]^2+4\sin^2 2\phi\}$$
$$\times\{[(\hat{q}^2+4\sin^2\phi)x+\hat{q}(\hat{q}^2+4)^{1/2}]^2+4\sin^2 2\phi\}/(\hat{q}^2+4\sin^2\phi)^2 \quad \text{(A15)}$$

the integral can be easily done giving

$$K = \frac{1}{2}\left(\mu^2 - m^2\sin^2\phi + \frac{q^2\cos 2\phi}{4}\right)$$
$$\times\arctan\left[\frac{\mu(\mu^2-m^2)^{1/2}\sin 2\phi}{\mu^2\cos 2\phi + m^2\sin^2\phi + q^2/4}\right]$$
$$-\frac{(q^2+2m^2)\sin 2\phi}{4}\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m} - \frac{q(q^2+4m^2)^{1/2}\sin 2\phi}{16}$$
$$\times\ln\frac{2\mu^2(2m^2+q^2)-2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)}{2\mu^2(2m^2+q^2)-2\mu q(q^2+4m^2)^{1/2}(\mu^2-m^2)^{1/2}-m^2(q^2+4m^2\sin^2\phi)} \quad \text{(A16)}$$

Inserting the results from equations (A7), (A11), (A13), and (A16) into equations (A1), (A2) we obtain the formulas (53), (54) for $\Delta\Pi_{00}$ and $\Delta\Pi^{\mu}_{\mu}$.

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